



NPTEL ONLINE CERTIFICATION COURSES

FINITE ELEMENT METHOD

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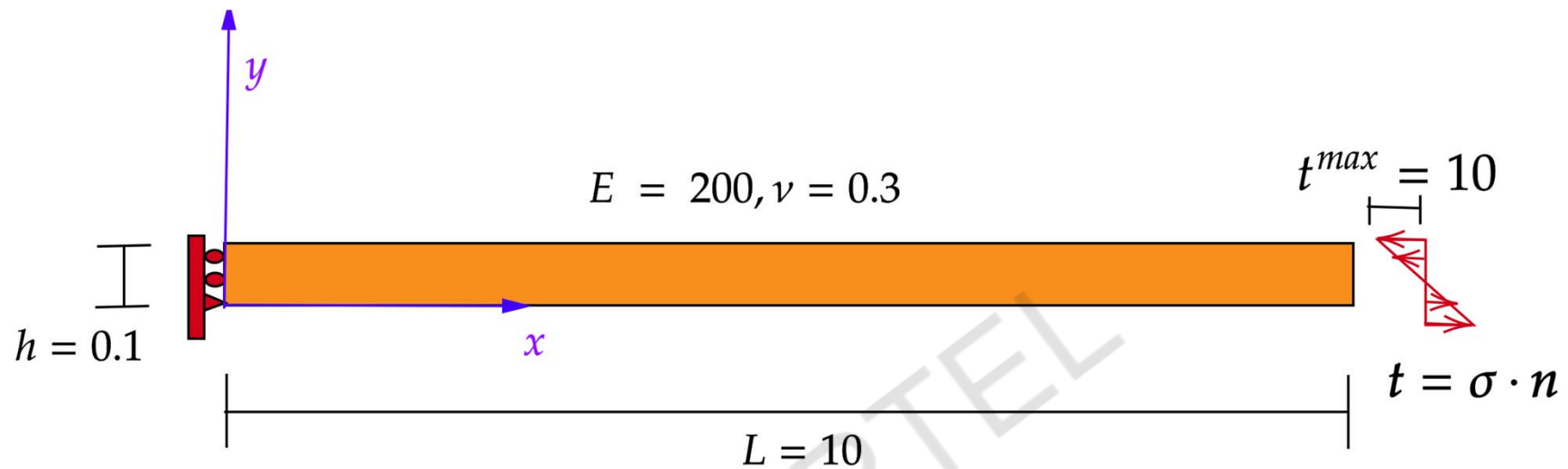
Module 12: Examples

Lecture 57 : B-Bar Method

Geometric Stiffness

- Incompatible Mode elements QM6 Example
- B-Bar method – Examples

Problem 1 : Thin Cantilever Plane stress Beam



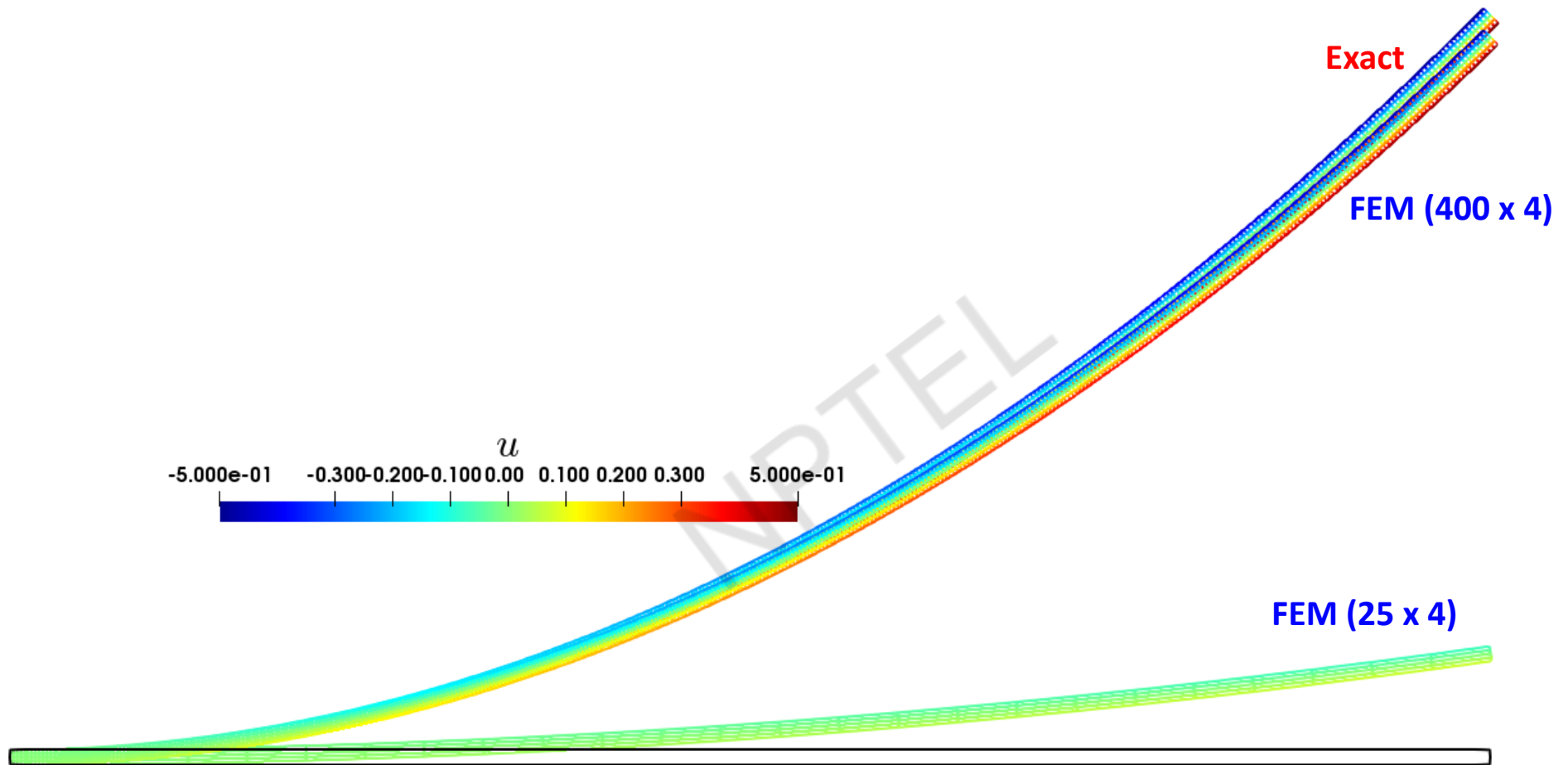
Thin Cantilever subjected to a couple at the free end $\frac{h}{L} = 0.01$

Exact Solution (Plane stress) :

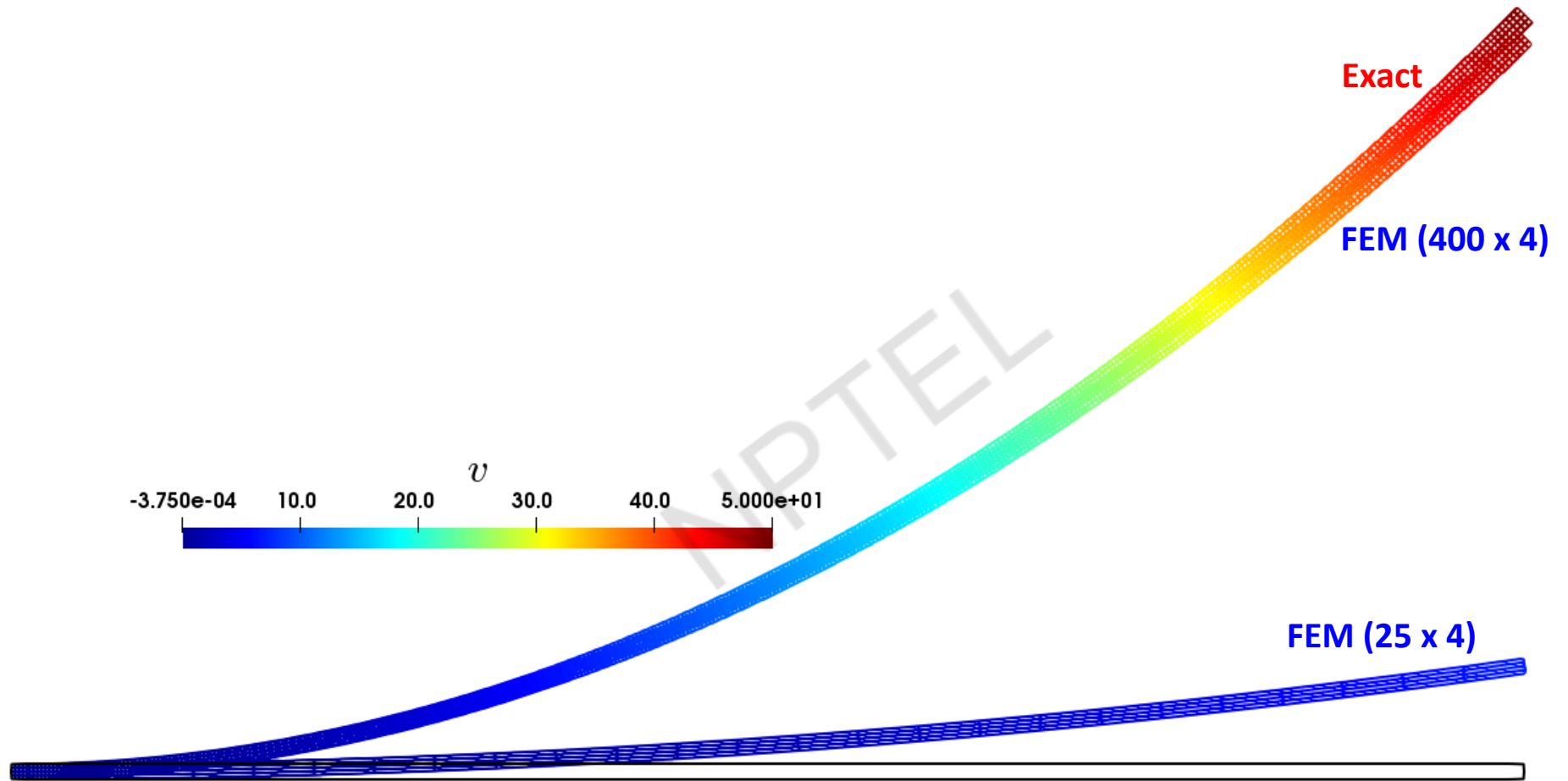
$$u(x, y) = \frac{2t^{\max}}{Eh} x \left(\frac{h}{2} - y \right)$$

$$v(x, y) = \frac{2t^{\max}}{Eh} (x^2 + \nu y(y - h))$$

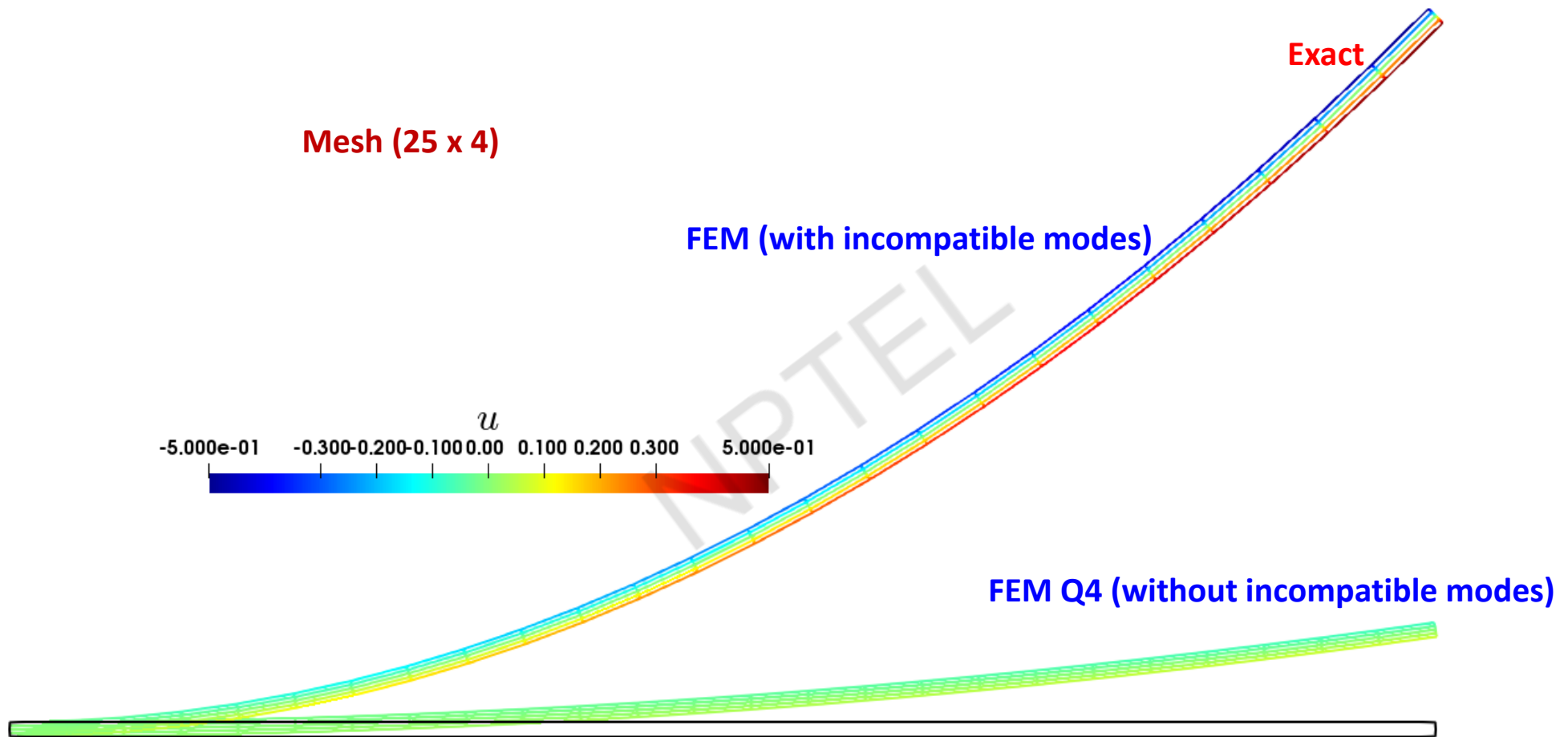
Problem 1 : Thin Cantilever Plane stress Beam



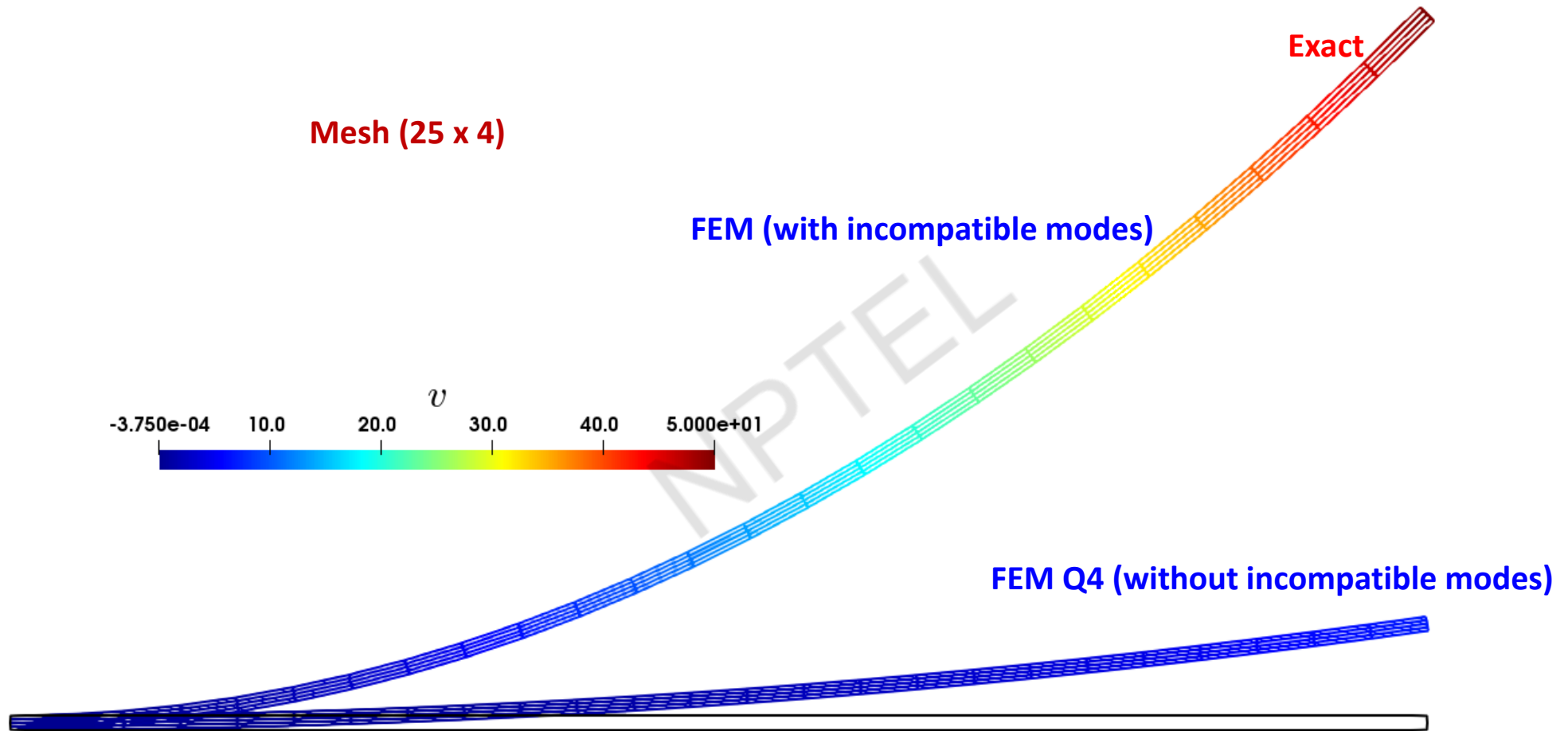
Problem 1 : Thin Cantilever Plane stress Beam



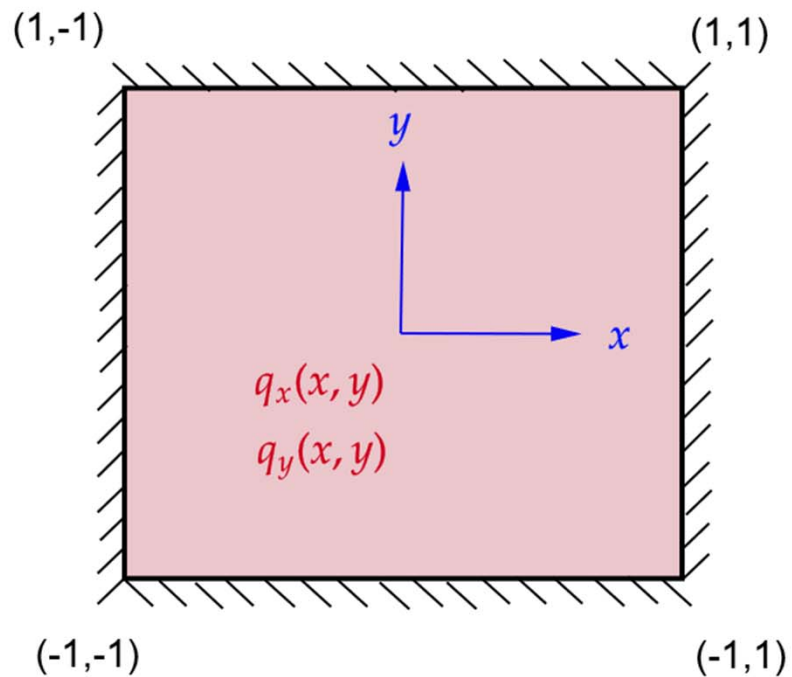
Problem 1 : Thin Cantilever Plane stress Beam



Problem 1 : Thin Cantilever Plane stress Beam



Problem 1 : Fully constrained 2D block

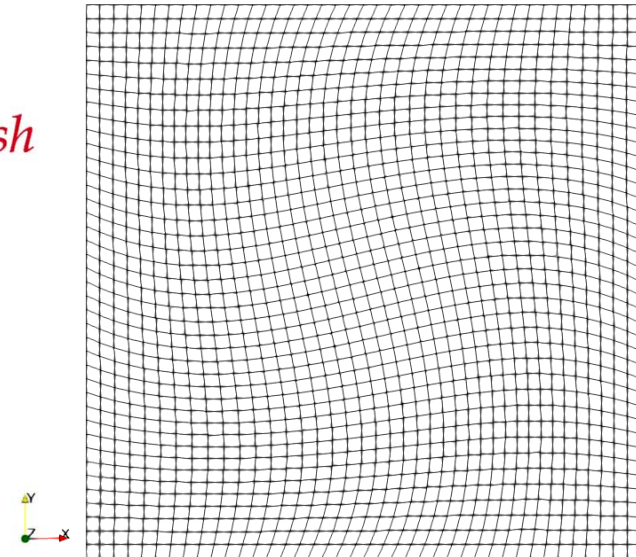


λ, μ – material parameters, $\mu = 40$

Exact Solution in 40×40 mesh

$$u(x, y) = \frac{1}{4} (x^2 - 1)^2 (y^2 - 1) y$$

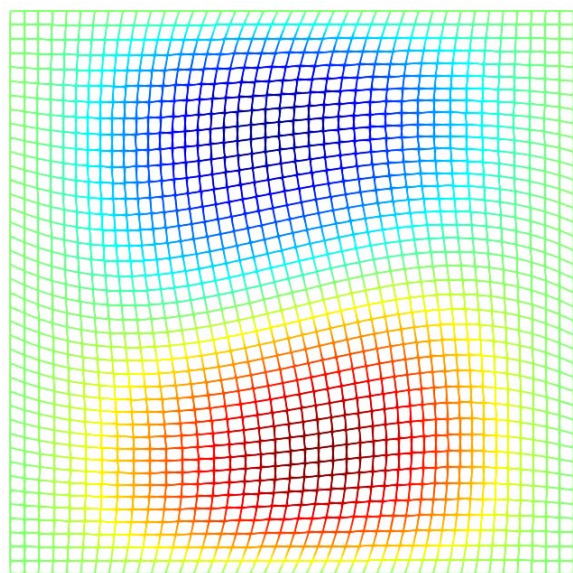
$$v(x, y) = \frac{1}{4} (1 - x^2) (y^2 - 1)^2 x$$



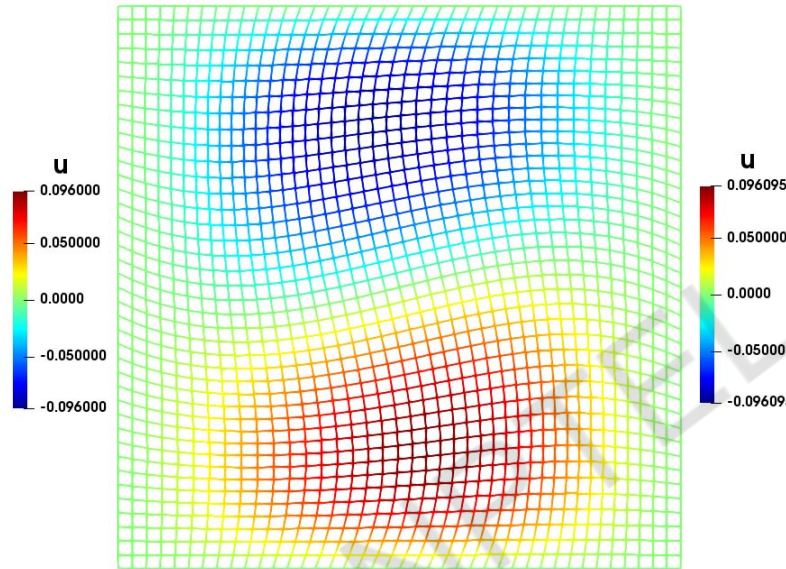
$$q_x(x, y) = \mu y (-1.5x^4 + 6x^2 - 3x^2y^2 + y^2 - 2.5) - 15x^2(y - 1)$$

$$q_y(x, y) = \mu x (1.5y^4 - 6y^2 + 3x^2y^2 - x^2 + 2.5) - 5x^3 - 3y^2$$

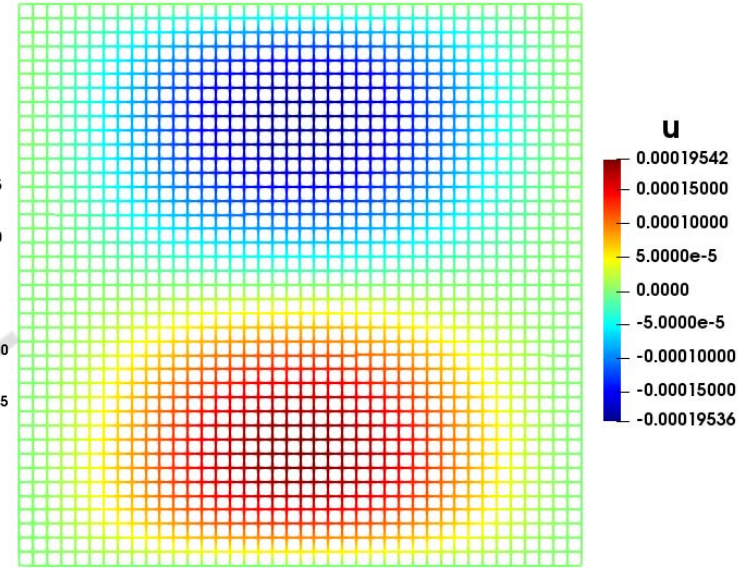
Problem 1 : Fully constrained 2D block



Exact



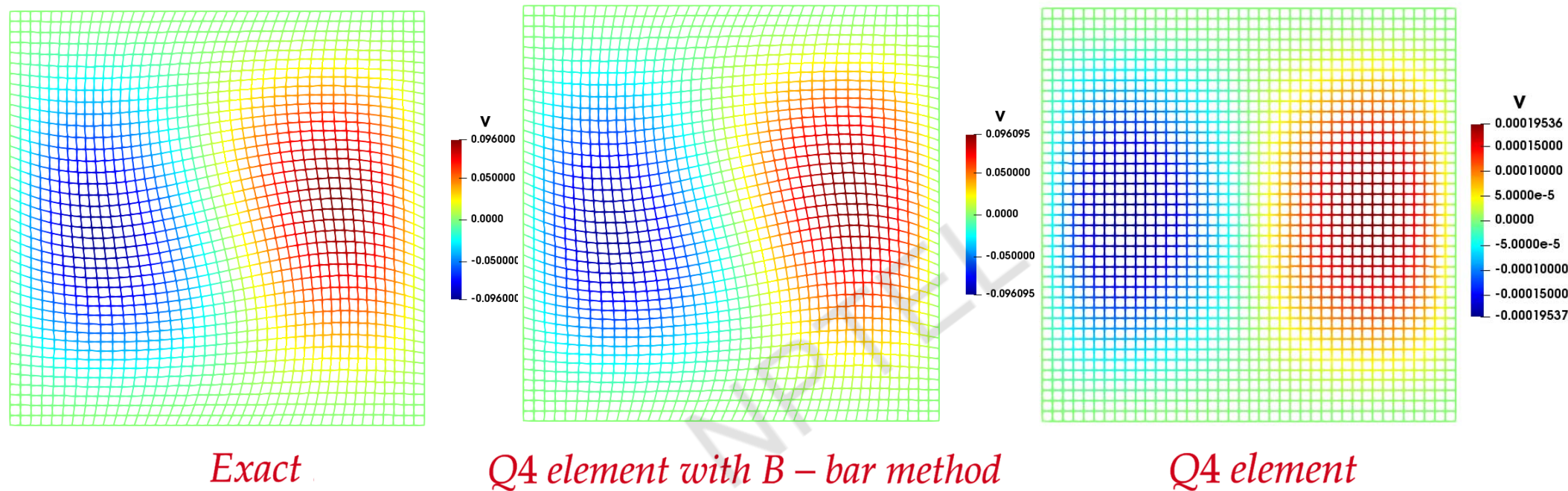
Q4 element with B – bar method



Q4 element

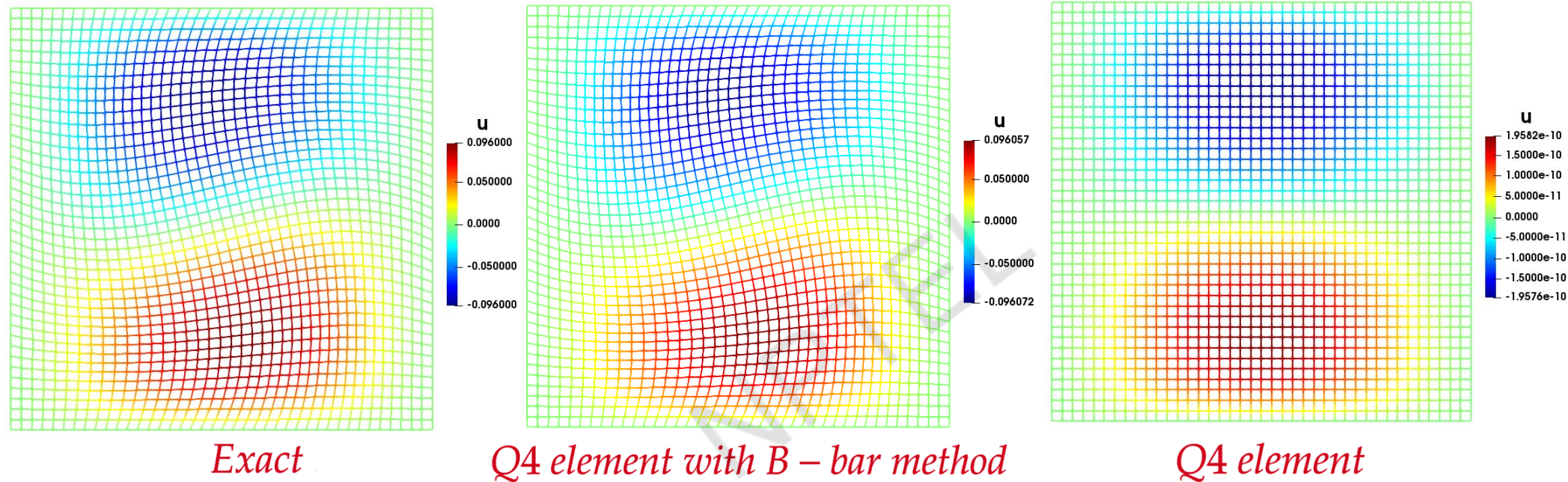
u displacement with $\frac{\lambda}{\mu} = 10^6$, $\nu = 0.499999$

Problem 1 : Fully constrained 2D block



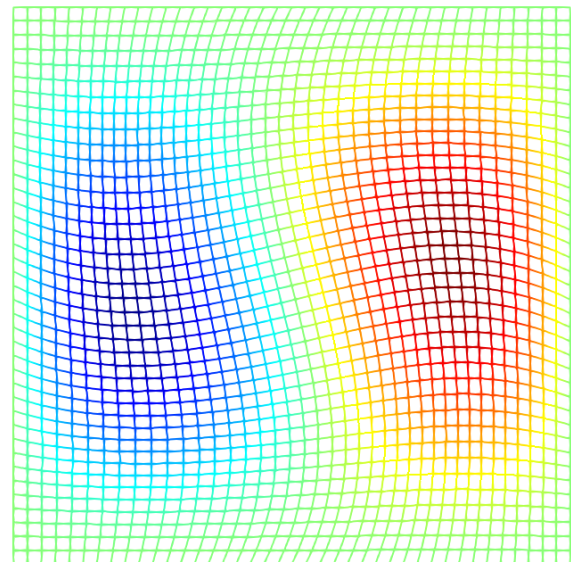
v displacement with $\frac{\lambda}{\mu} = 10^6$, $\nu = 0.499999$

Problem 1 : Fully constrained 2D block

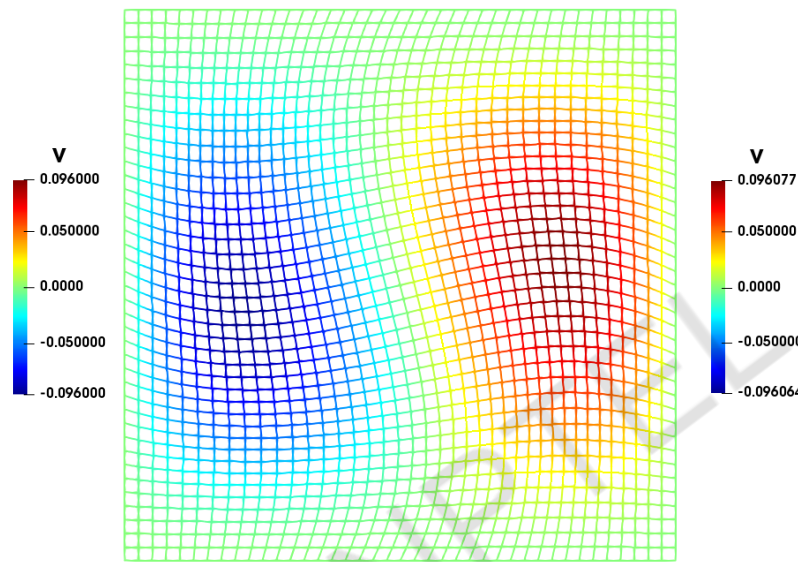


u displacement with $\frac{\lambda}{\mu} = 10^{12}$, $\nu = 0.5 - 10^{-12}$

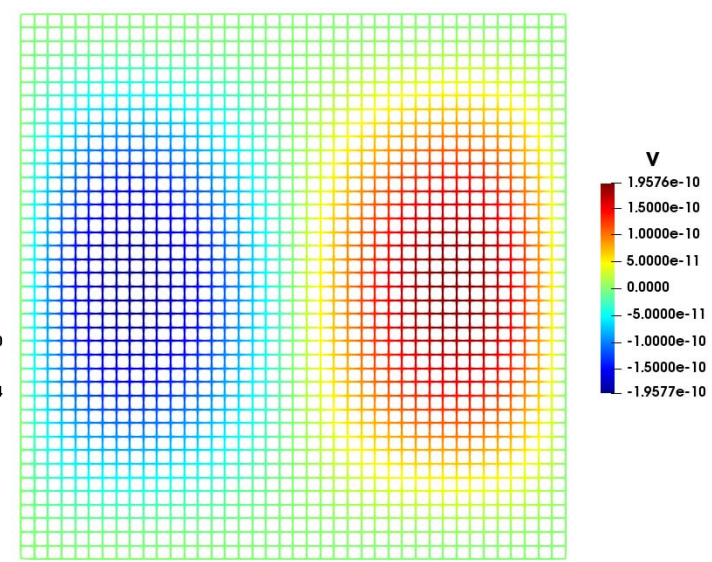
Problem 1 : Fully constrained 2D block



Exact



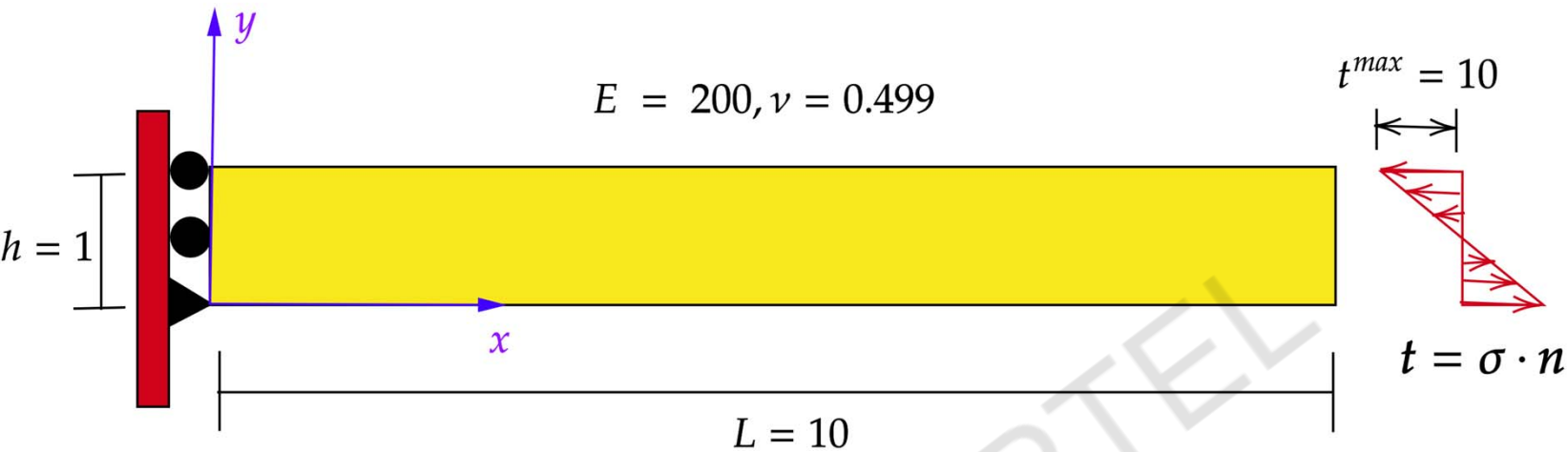
Q4 element with B – bar method



Q4 element

v displacement with $\frac{\lambda}{\mu} = 10^{12}$, $\nu = 0.5 - 10^{-12}$

Problem 2: Nearly incompressible cantilever



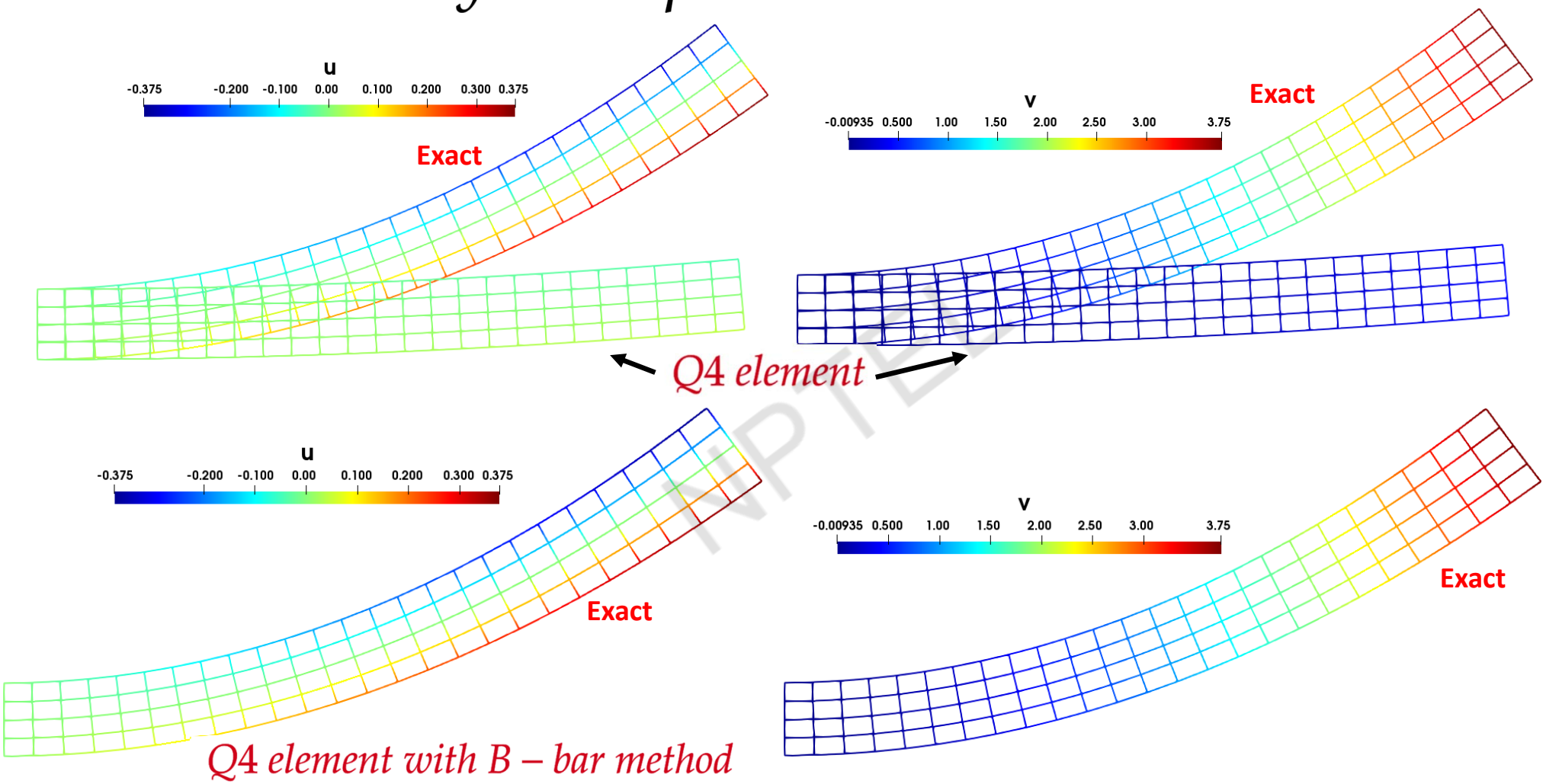
Simply Supported Cantilever subjected to a couple at the free end

Exact Solution :

$$u(x, y) = \frac{2t^{\max}(1-\nu^2)}{Eh}x\left(\frac{h}{2} - y\right)$$

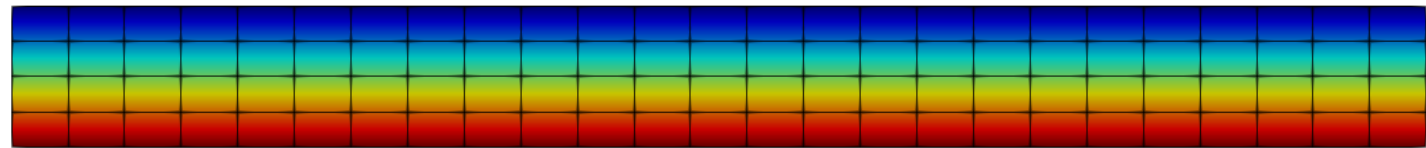
$$v(x, y) = \frac{2t^{\max}(1-\nu^2)}{Eh}\left(x^2 + \frac{\nu}{1-\nu}y(y-h)\right)$$

Problem 2: Nearly incompressible cantilever

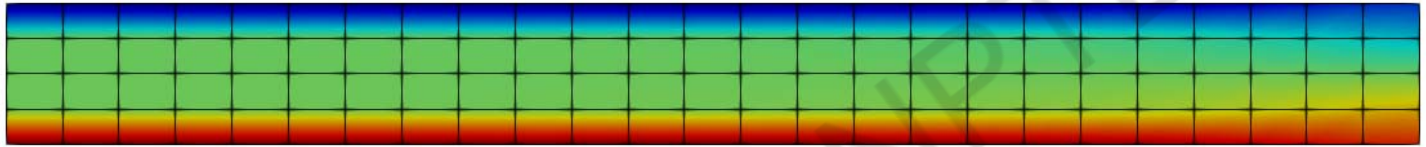


Problem 2: Nearly incompressible cantilever

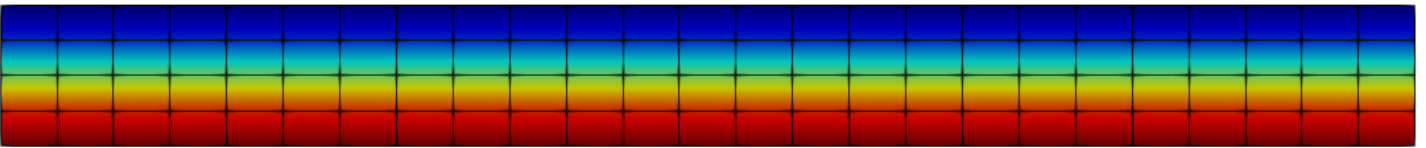
Exact E_v



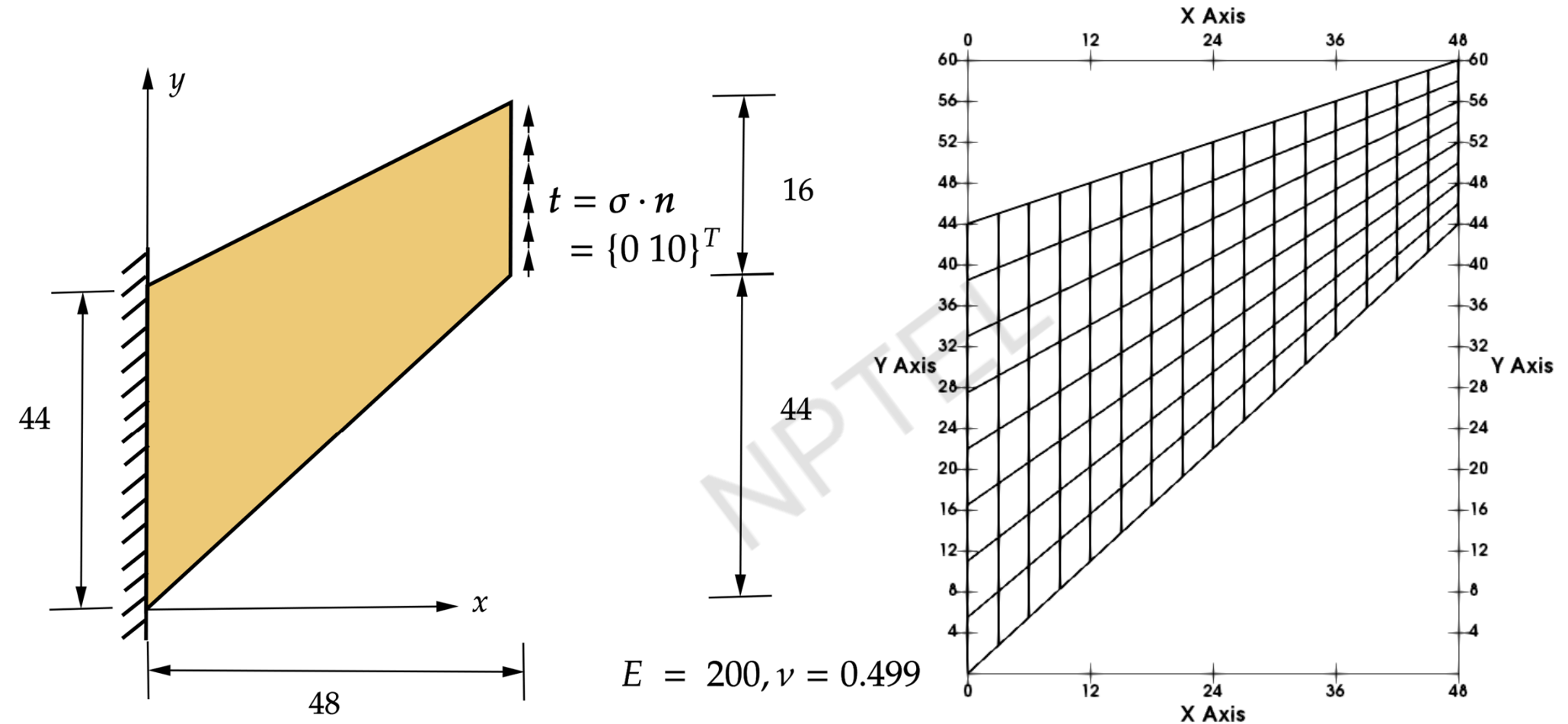
E_v



E_v with B-Bar



Problem 3 : Cook's Membrane



Cook – type cantilever beam

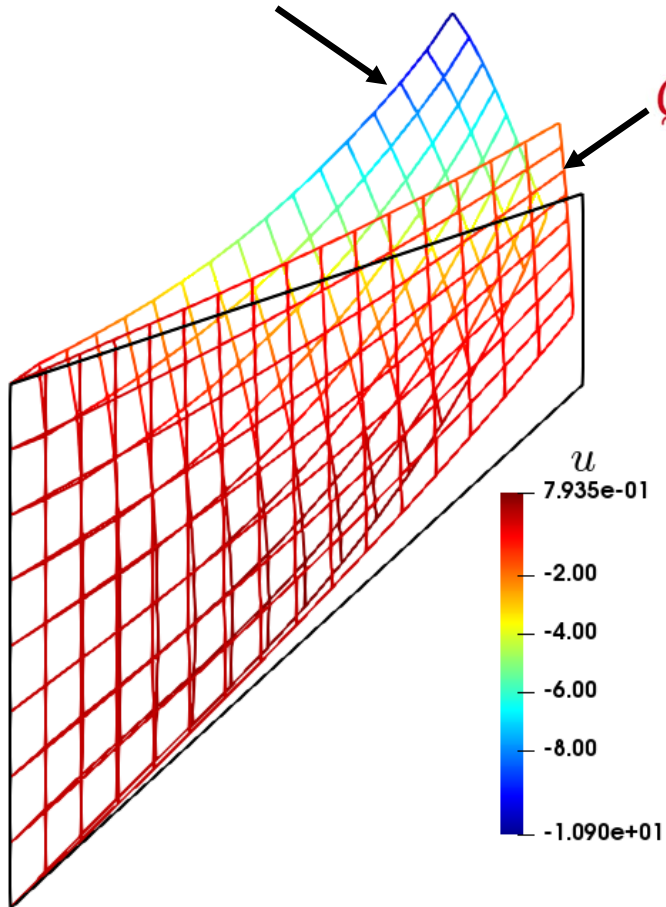
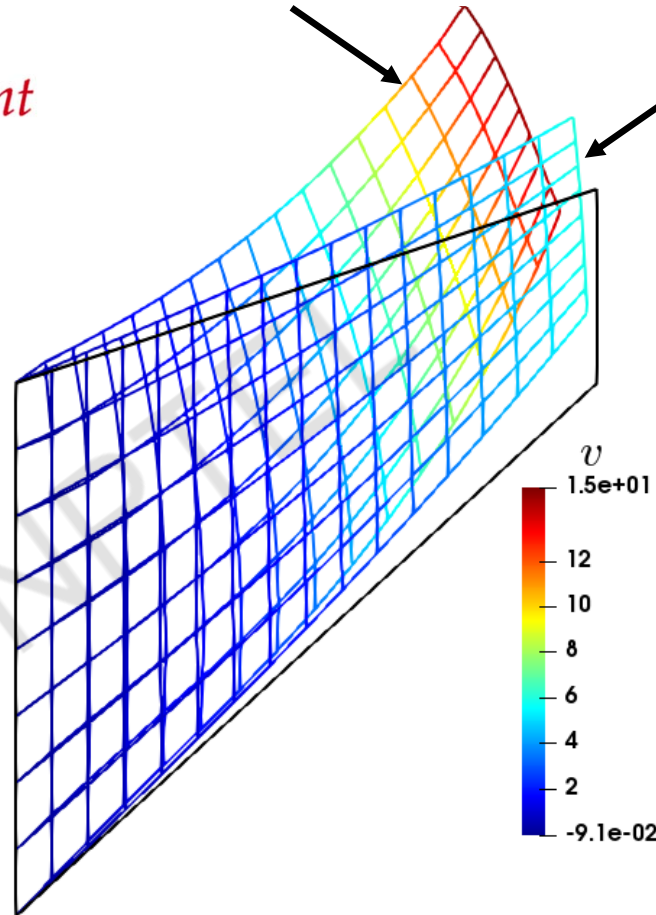
Problem 3 : Cook's Membrane

Q4 element with B – bar method

Q4 element with B – bar method

Q4 element

Q4 element



$$E = 200, \nu = 0.499$$

Problem 3 : Cook's Membrane

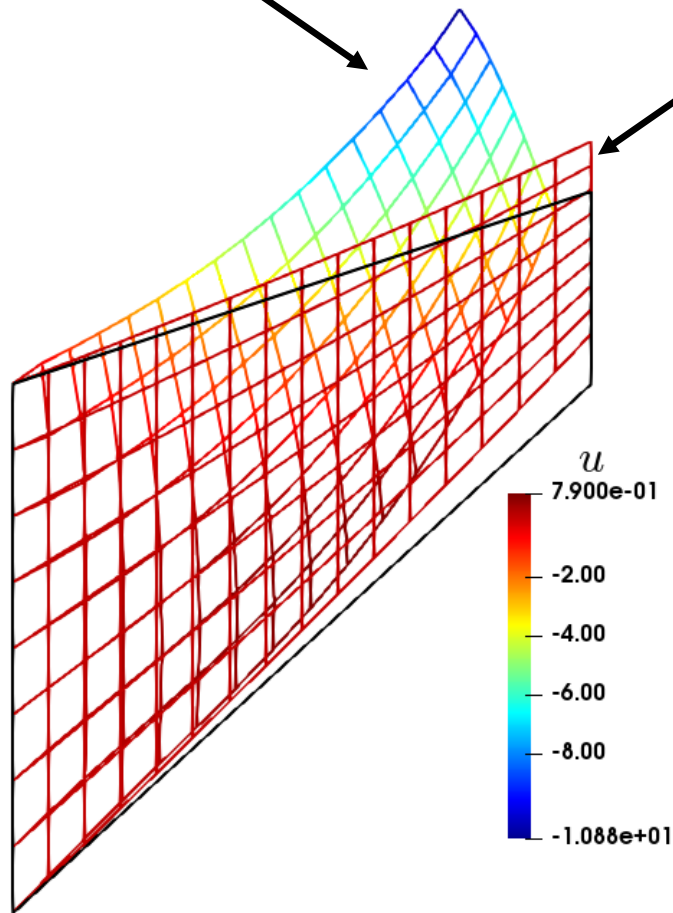
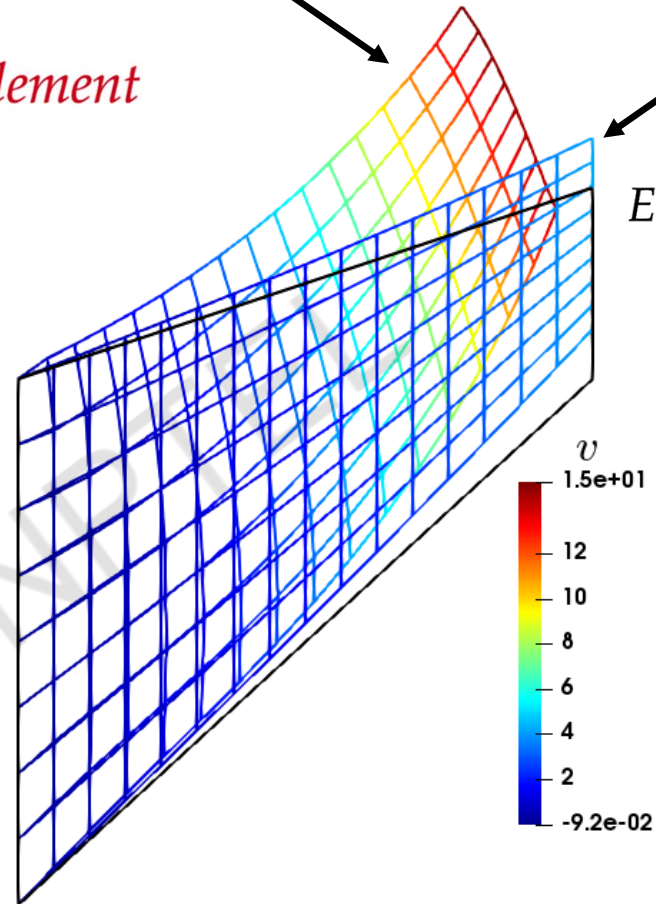
Q4 element with B – bar method

Q4 element with B – bar method

Q4 element

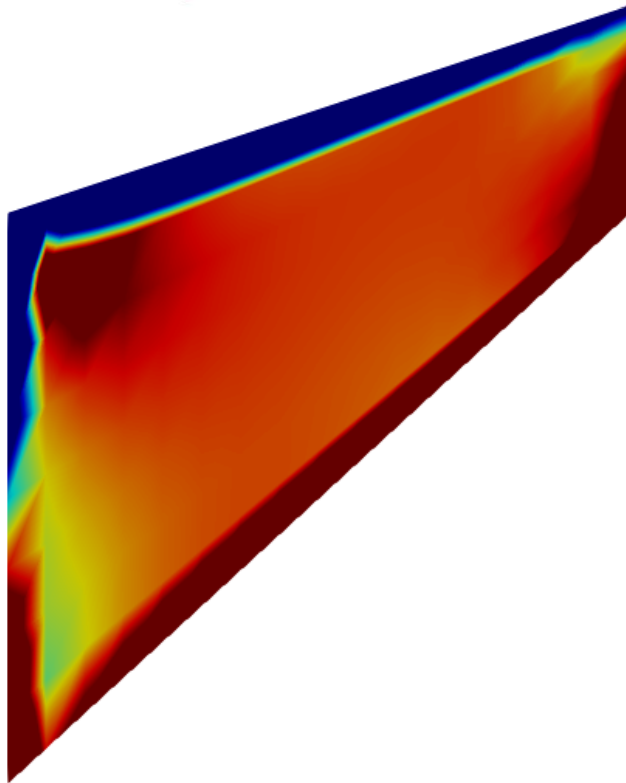
Q4 element

$E = 200, \nu = 0.499999$

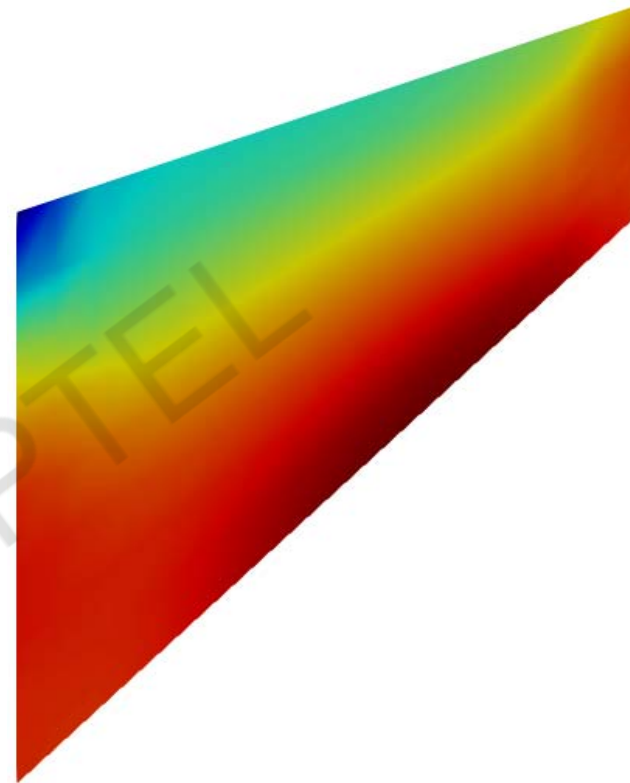


Problem 3 : Cook's Membrane

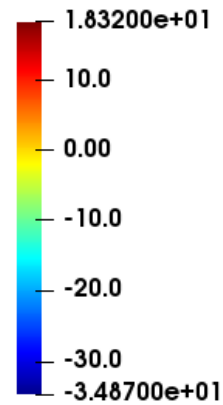
Q4 element



Q4 element with B – bar method



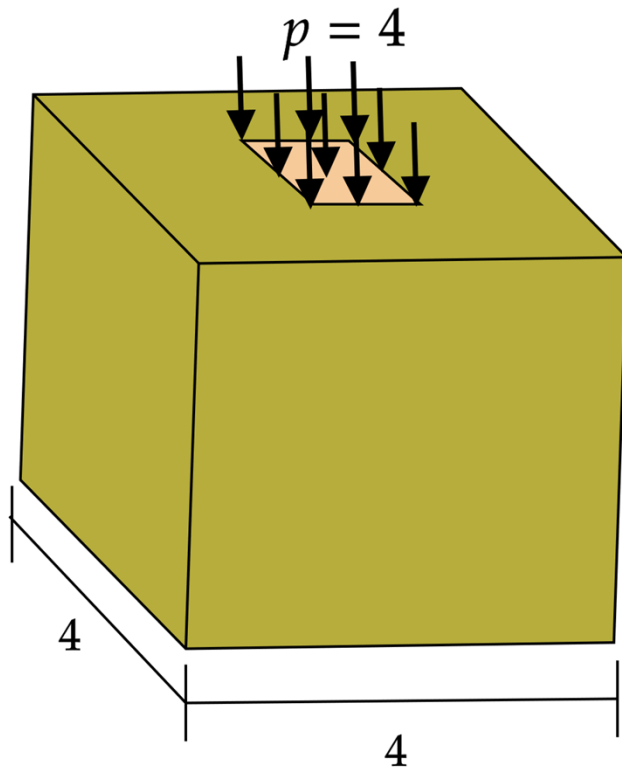
Pressure



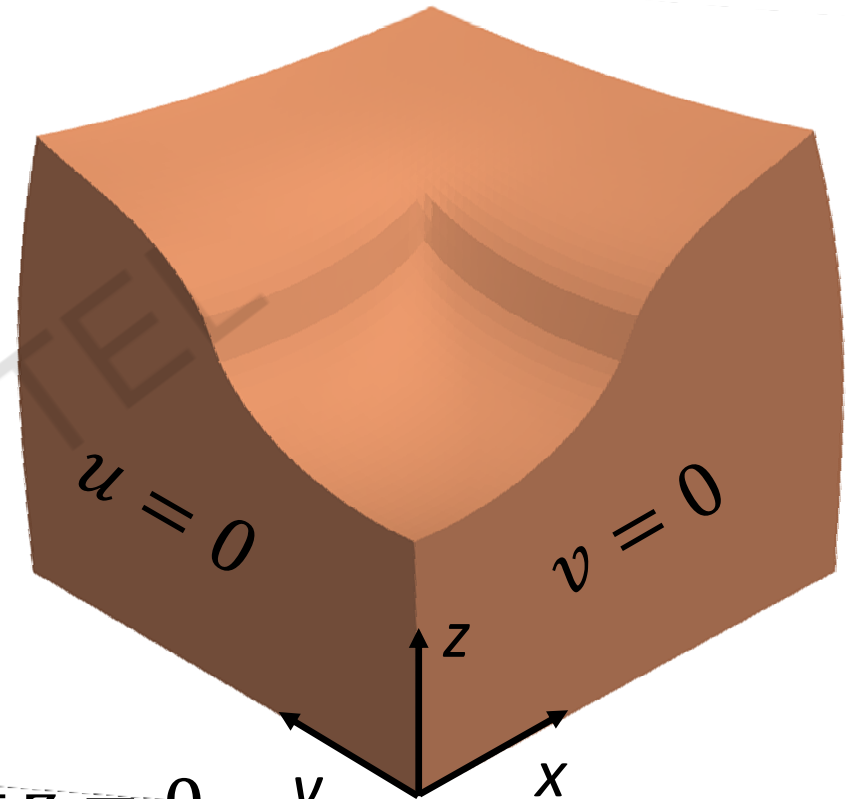
$$E = 200, \nu = 0.499999$$

Problem 4: Nearly incompressible 3D block under compression

$$u = 0 \text{ and } v = 0 \text{ at } z = 1$$



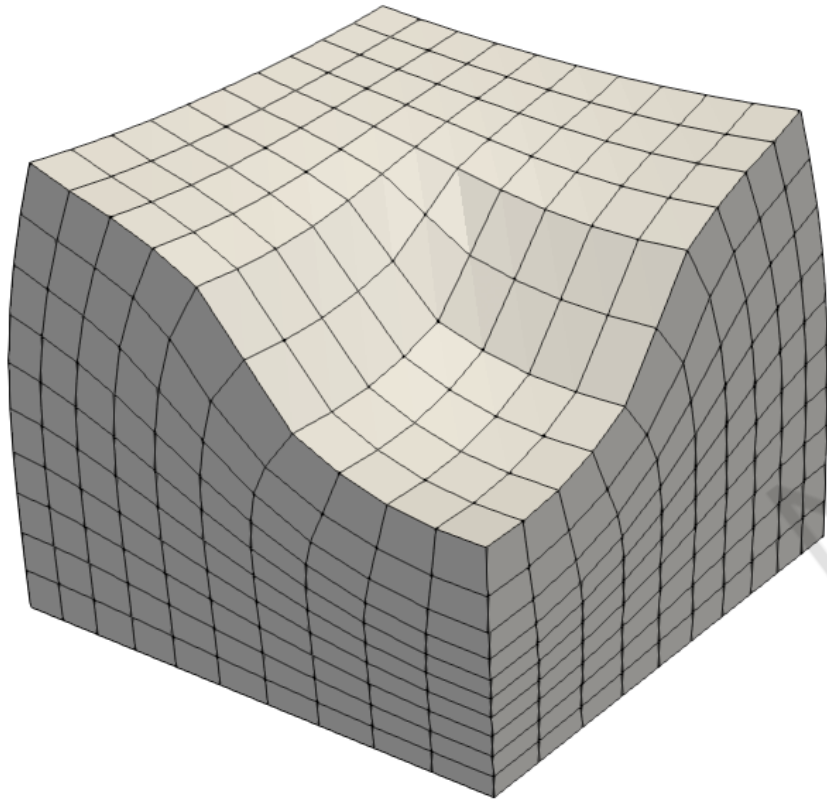
$$E = 240.57$$
$$\nu = 0.4999$$



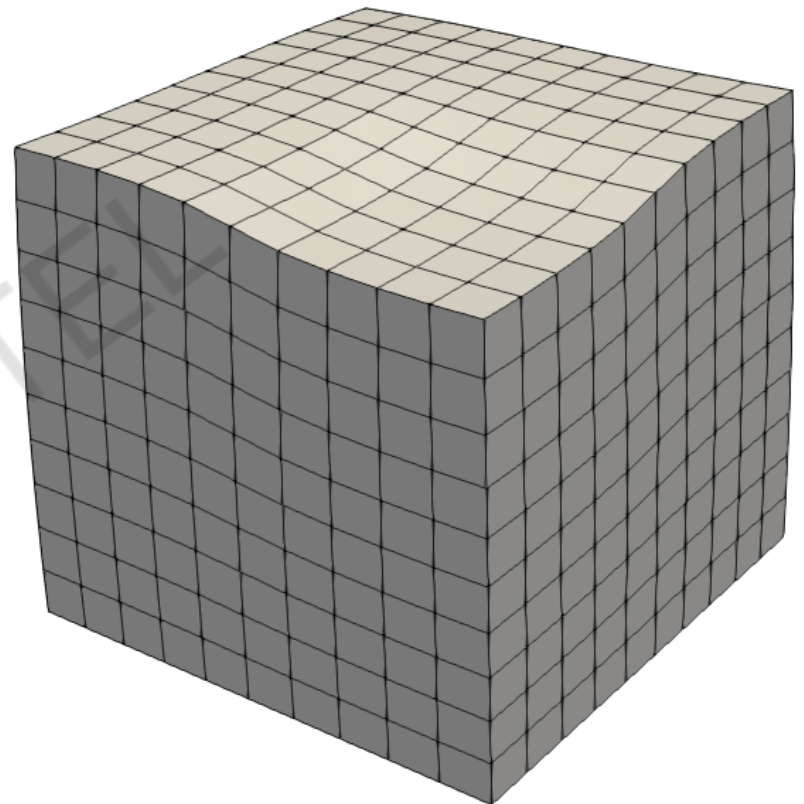
$$w = 0 \text{ at } z = 0$$

Problem 4: Nearly incompressible 3D block under compression

P8 element with B – bar

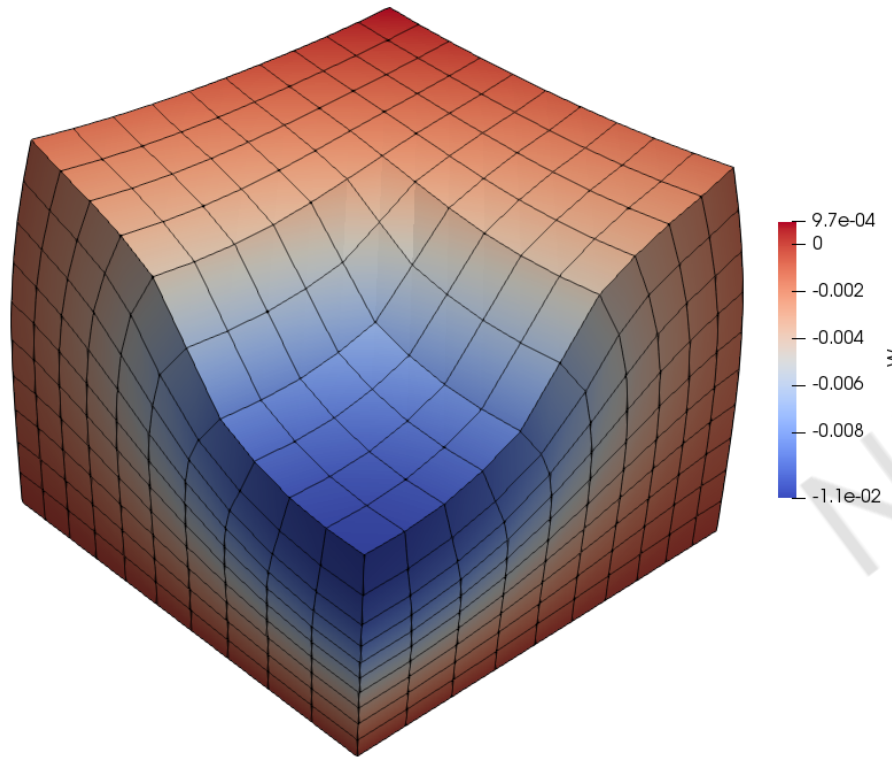


P8 element

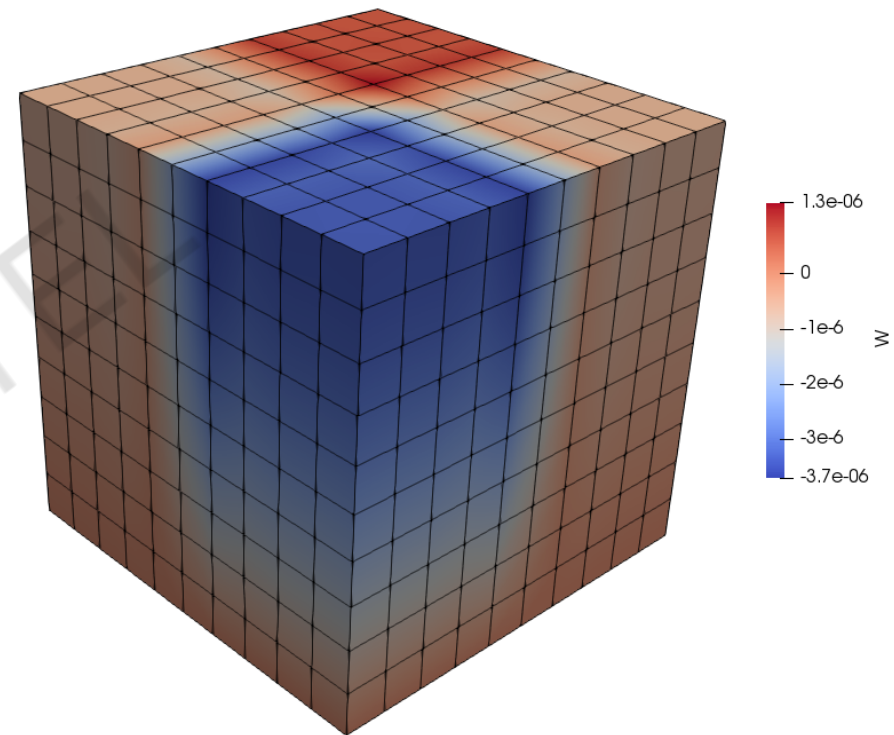


Problem 4: Nearly incompressible 3D block under compression

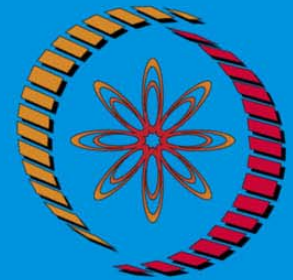
P8 element with B – bar



P8 element



$$\nu = 0.499999899979815$$



**THANK
YOU !**



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Finite Element Method

Week 12: Finite element in 3D

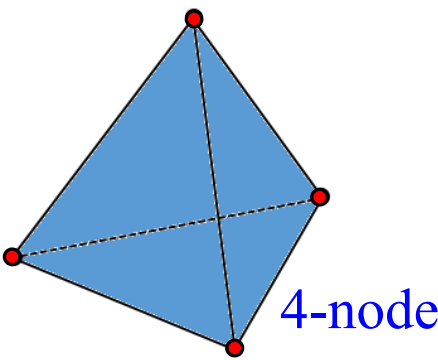
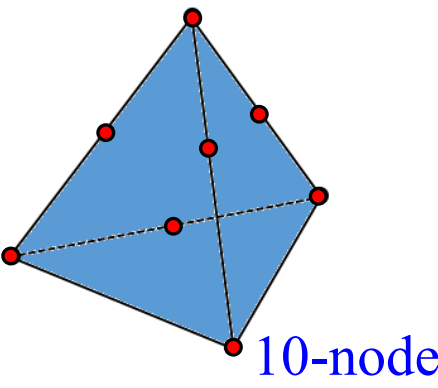
Lecture 58: Different Elements

Biswanath Banerjee
Associate Professor
Department of Civil Engineering
IIT Kharagpur

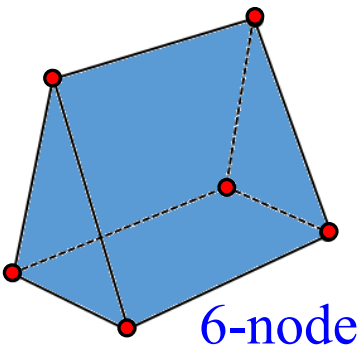
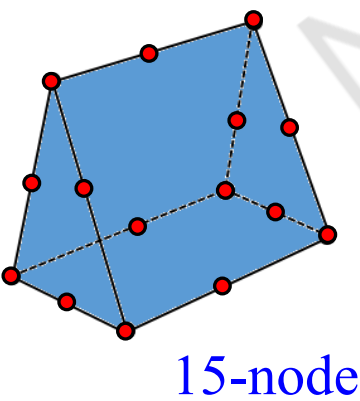
Amit Shaw
Associate Professor
Department of Civil Engineering
IIT Kharagpur

Different Finite Elements in Three Dimension

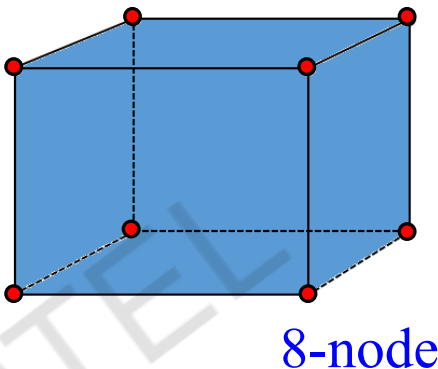
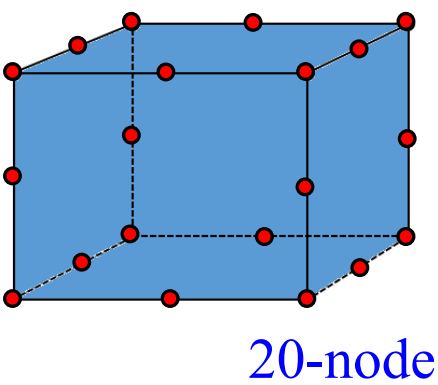
Tetra



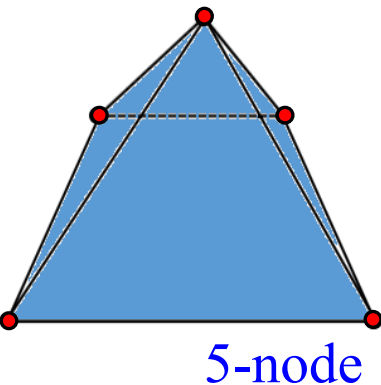
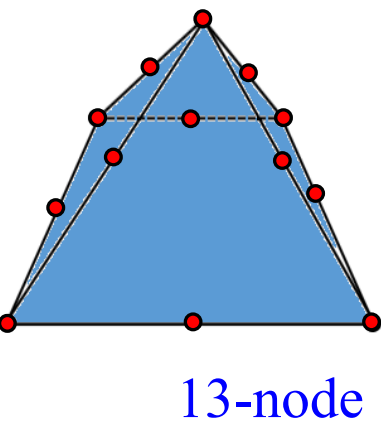
Wedge/ Penta



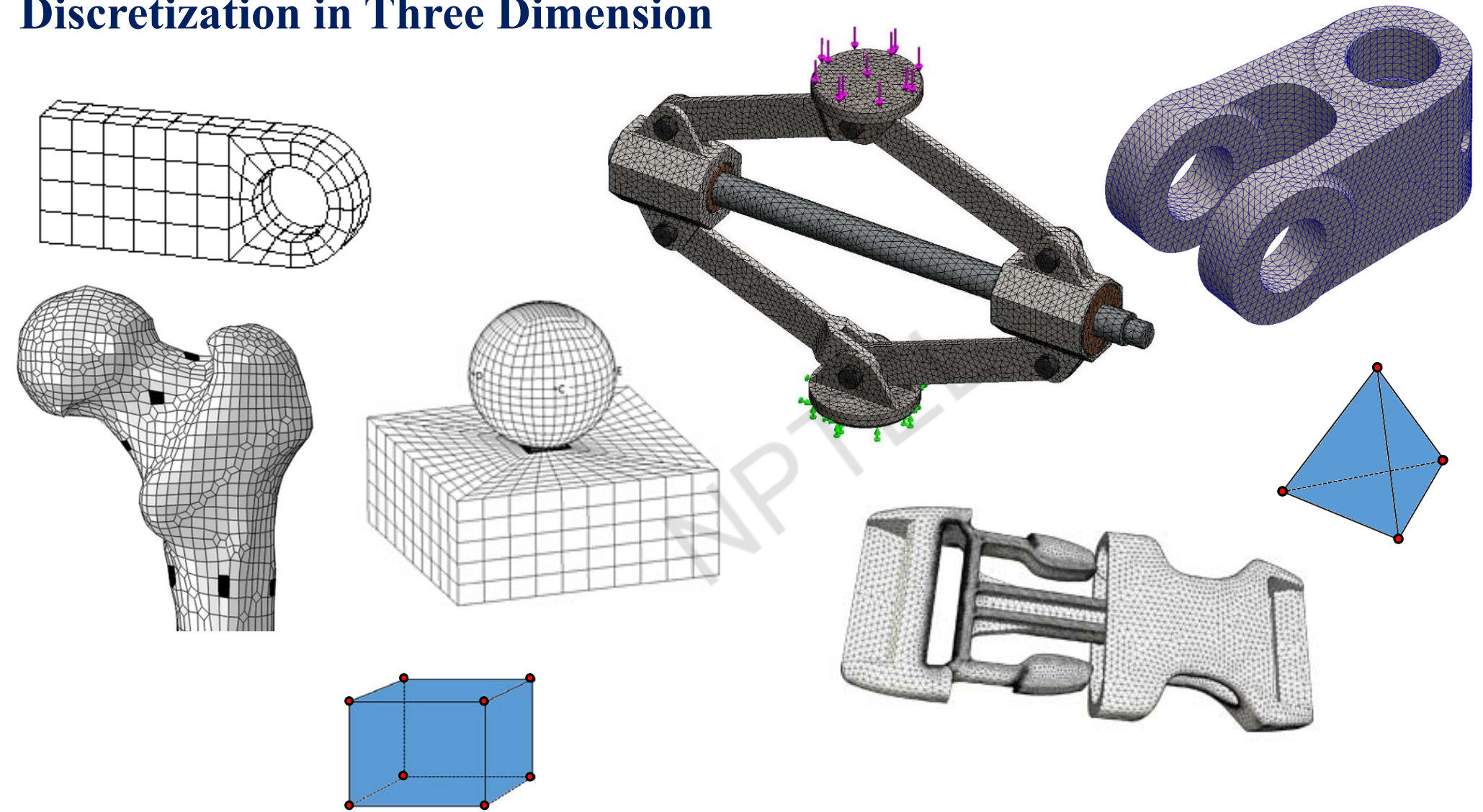
Hex/ Brick



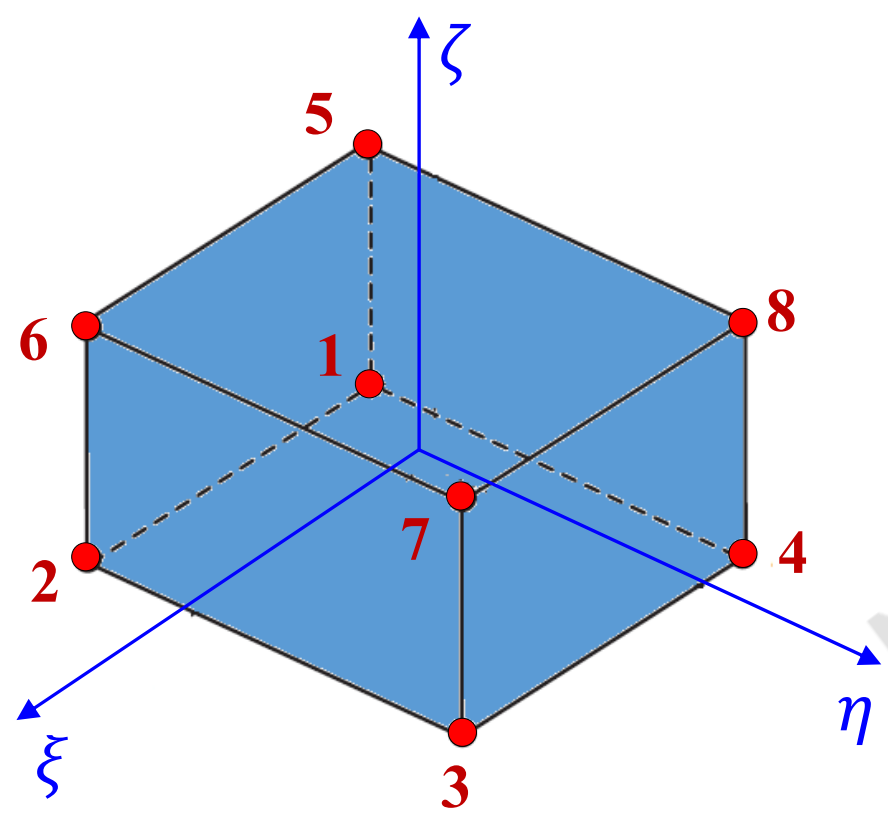
Pyramid



Discretization in Three Dimension



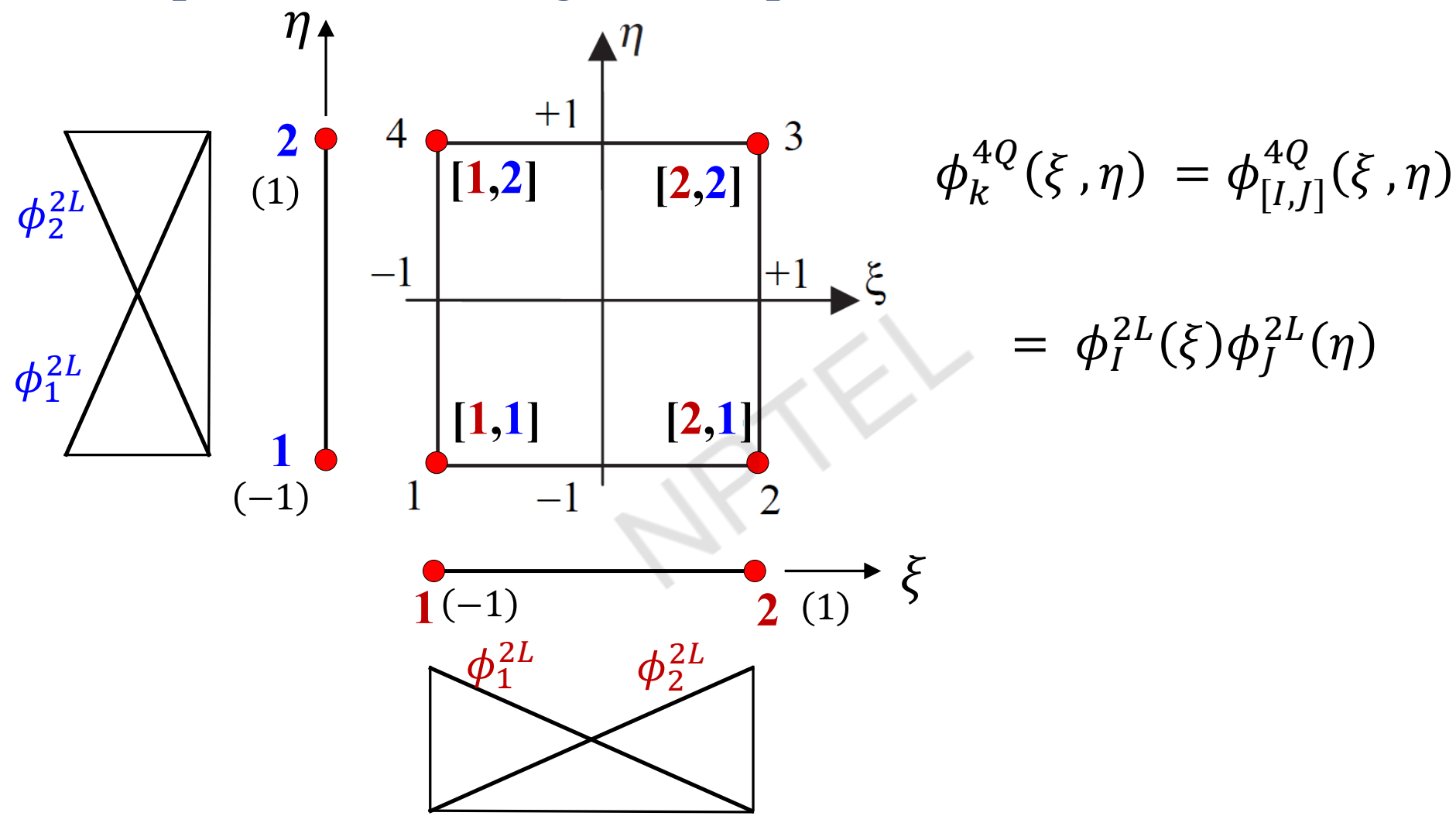
Shape function



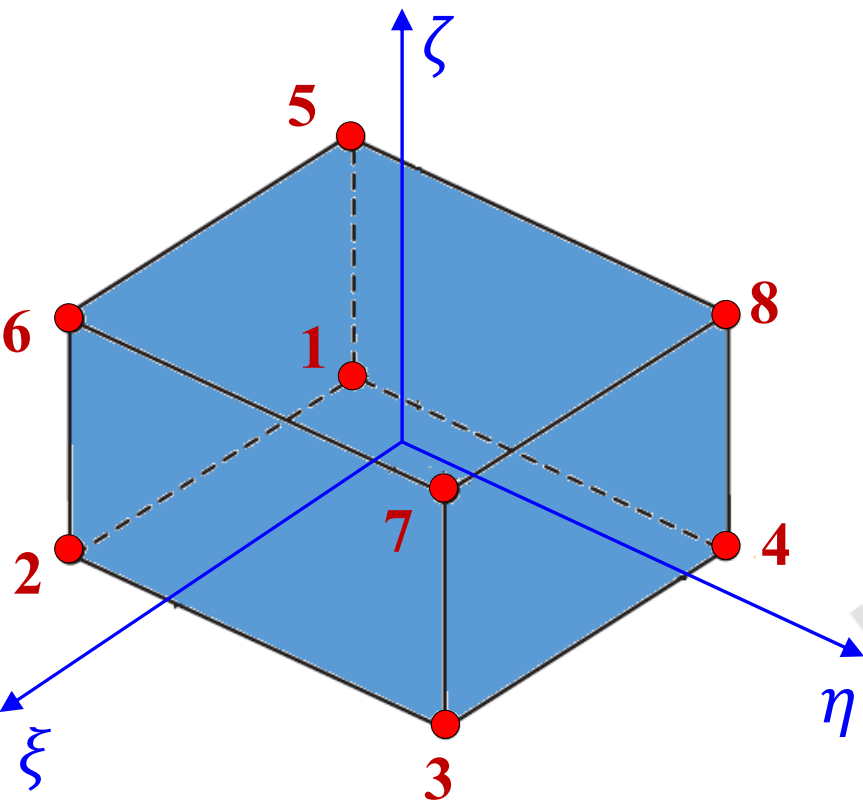
$[-1,1] \times [-1,1] \times [-1,1]$

NPTTEL

Recall: Shape function through tensor product



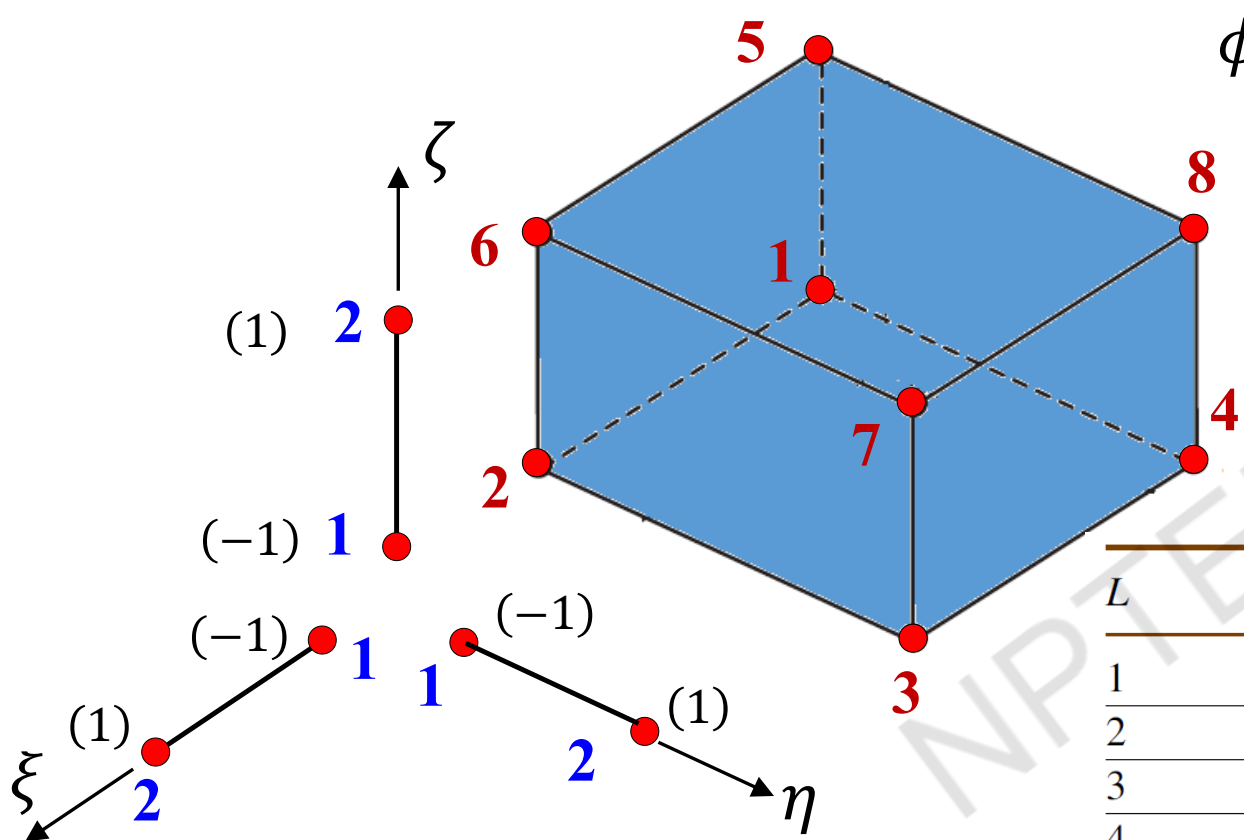
Shape function: 8-node hexahedral (or brick) element



$$[-1,1] \times [-1,1] \times [-1,1]$$

$$\begin{aligned} \phi_L^{8H}(\xi, \eta, \zeta) &= \phi_{[I,J,K]}^{8H}(\xi, \eta, \zeta) \\ &= \phi_I^{2L}(\xi) \phi_J^{2L}(\eta) \phi_K^{2L}(\zeta) \end{aligned}$$

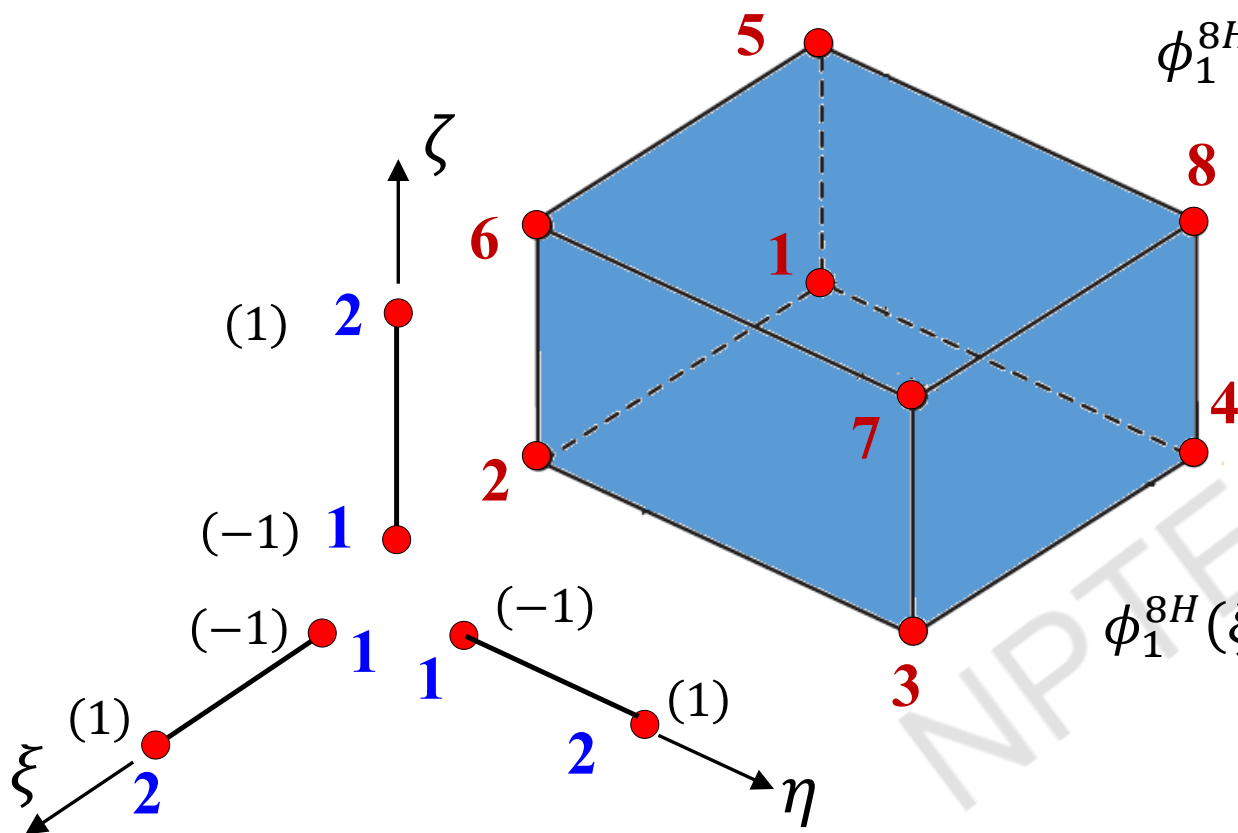
L	I	J	K
1	1	1	1
2	2	1	1
3	2	2	1
4	1	2	1
5	1	1	2
6	2	1	2
7	2	2	2
8	1	2	2



$$\phi_L^{8H}(\xi, \eta, \zeta) = \phi_{[I,J,K]}^{8H}(\xi, \eta, \zeta)$$

$$= \phi_I^{2L}(\xi) \phi_J^{2L}(\eta) \phi_J^{2L}(\zeta)$$

L	I	J	K
1	1	1	1
2	2	1	1
3	2	2	1
4	1	2	1
5	1	1	2
6	2	1	2
7	2	2	2
8	1	2	2



$$\phi_1^{8H}(\xi, \eta, \zeta) = \phi_1^{2L}(\xi)\phi_1^{2L}(\eta)\phi_1^{2L}(\zeta)$$

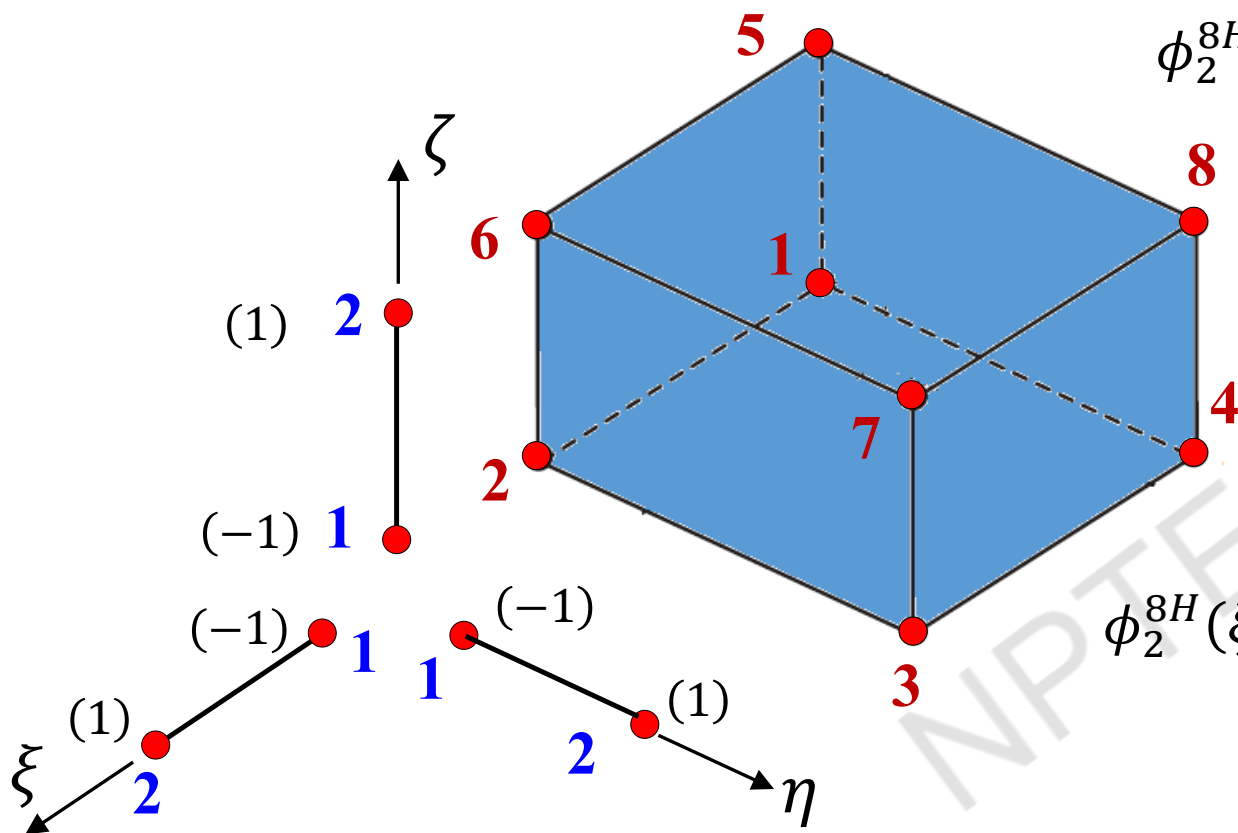
$$\phi_1^{2L}(\xi) = (1 - \xi)/2$$

$$\phi_1^{2L}(\eta) = (1 - \eta)/2$$

$$\phi_1^{2L}(\zeta) = (1 - \zeta)/2$$

$$\phi_1^{8H}(\xi, \eta, \zeta) = \frac{1}{8}(1 - \xi)(1 - \eta)(1 - \zeta)$$

L	I	J	K
1	1	1	1
2	2	1	1



$$\phi_2^{8H}(\xi, \eta, \zeta) = \phi_2^{2L}(\xi) \phi_1^{2L}(\eta) \phi_1^{2L}(\zeta)$$

$$\phi_2^{2L}(\xi) = (1 + \xi)/2$$

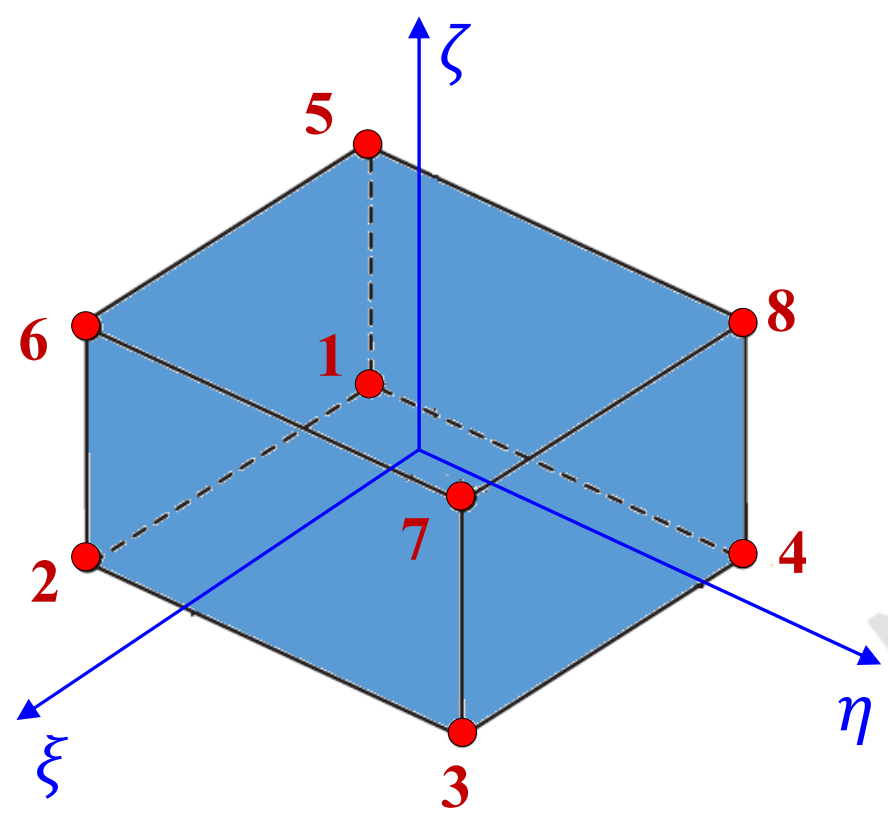
$$\phi_1^{2L}(\eta) = (1 - \eta)/2$$

$$\phi_1^{2L}(\zeta) = (1 - \zeta)/2$$

$$\phi_2^{8H}(\xi, \eta, \zeta) = \frac{1}{8} (1 + \xi)(1 - \eta)(1 - \zeta)$$

L	I	J	K
1	1	1	1
2	2	1	1

Shape function: 8-node hexahedral (or brick) element

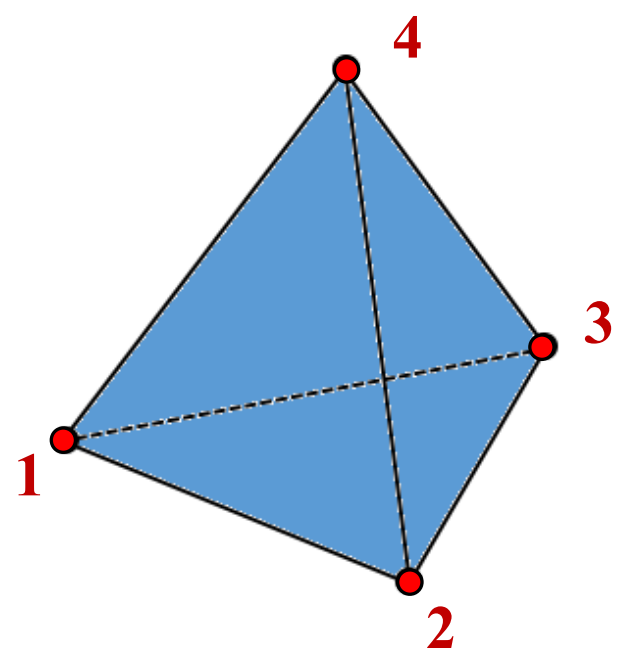


$$[-1,1] \times [-1,1] \times [-1,1]$$

$$\begin{aligned} \phi_1^{8H}(\xi, \eta, \zeta) &= (1 - \xi)(1 - \eta)(1 - \zeta)/8 \\ \phi_2^{8H}(\xi, \eta, \zeta) &= (1 + \xi)(1 - \eta)(1 - \zeta)/8 \\ \phi_3^{8H}(\xi, \eta, \zeta) &= (1 + \xi)(1 + \eta)(1 - \zeta)/8 \\ \phi_4^{8H}(\xi, \eta, \zeta) &= (1 - \xi)(1 + \eta)(1 - \zeta)/8 \\ \phi_5^{8H}(\xi, \eta, \zeta) &= (1 - \xi)(1 - \eta)(1 + \zeta)/8 \\ \phi_6^{8H}(\xi, \eta, \zeta) &= (1 + \xi)(1 - \eta)(1 + \zeta)/8 \\ \phi_7^{8H}(\xi, \eta, \zeta) &= (1 + \xi)(1 + \eta)(1 + \zeta)/8 \\ \phi_8^{8H}(\xi, \eta, \zeta) &= (1 - \xi)(1 + \eta)(1 + \zeta)/8 \end{aligned}$$

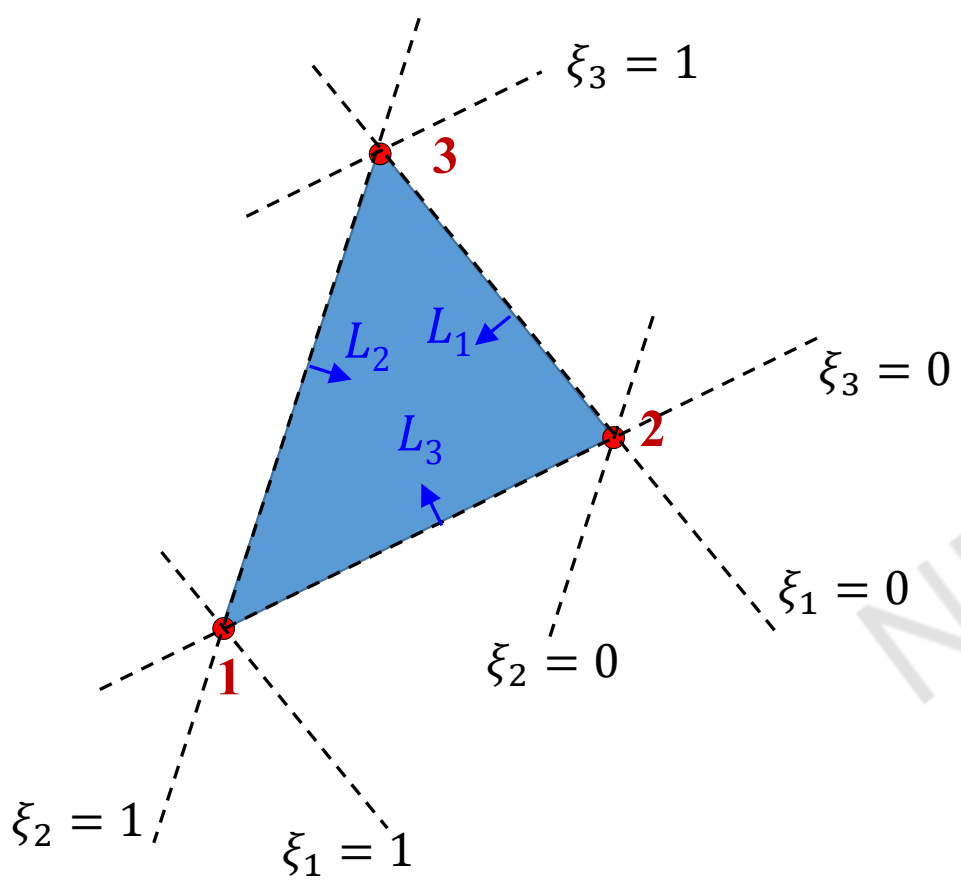
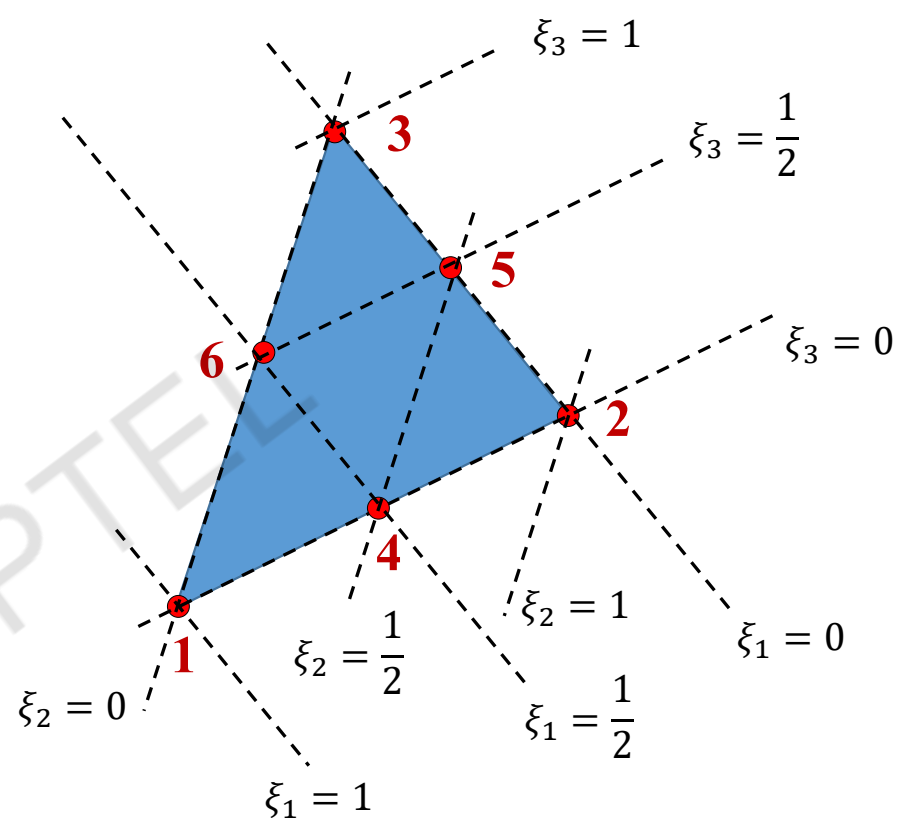
$$\phi_i^e(\xi, \eta) = \frac{1}{8} (1 + \xi_i \xi)(1 + \eta_i \eta)(1 + \zeta_i \zeta)$$

Shape function: 4-node tetrahedral element

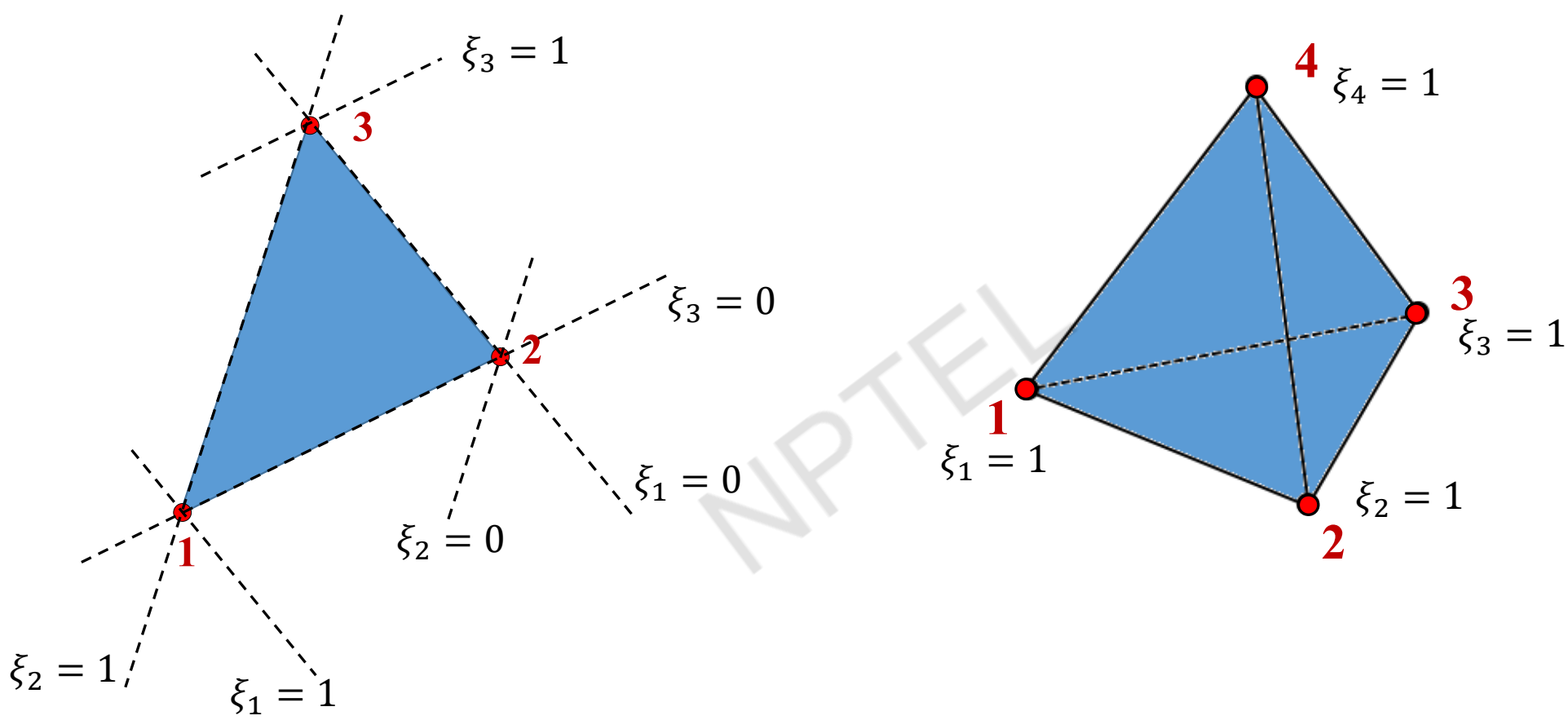


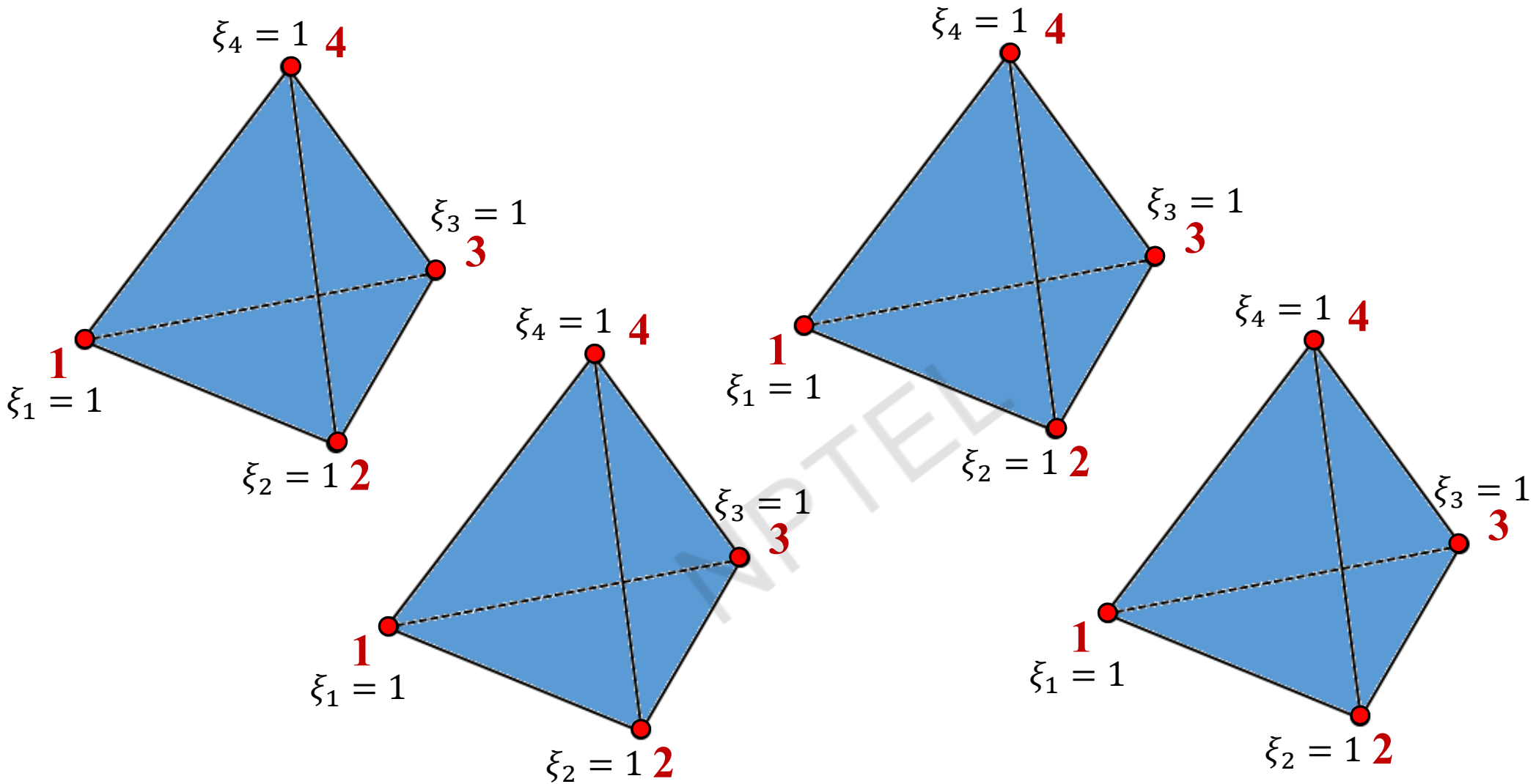
NPTEL

Recall: Area coordinate

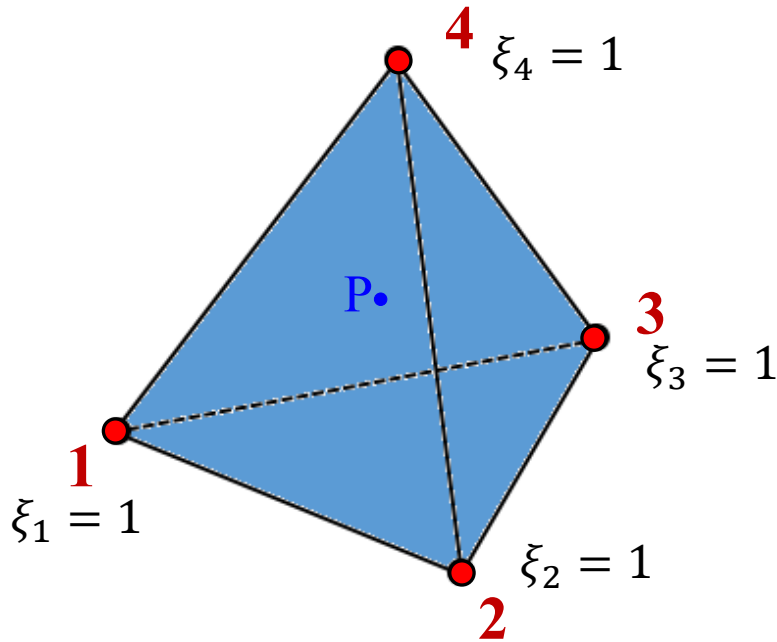


Shape function: 4-node tetrahedral element





Shape function: 4-node tetrahedral element



$$\xi_1 = \frac{\text{volume of } P234}{\Omega^e}$$

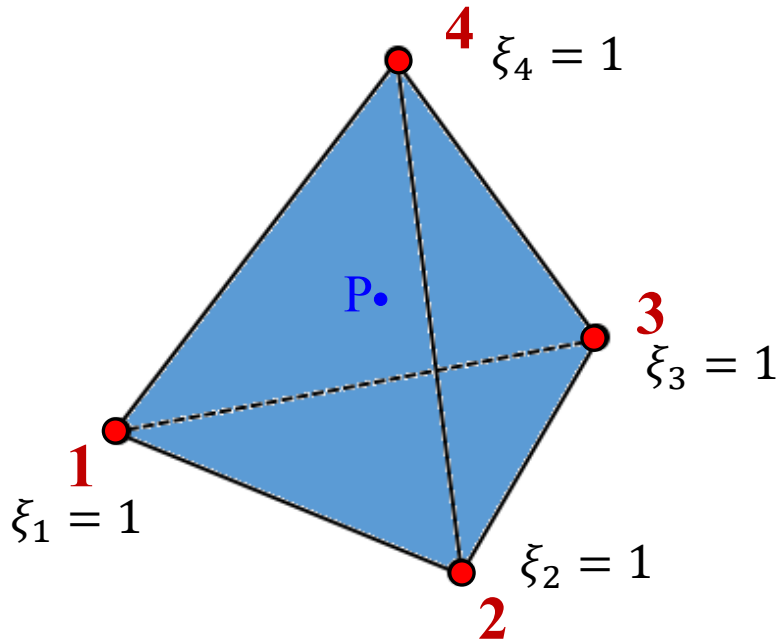
Similarly,

$$\xi_2 = \frac{\text{volume of } P134}{\Omega^e}$$

$$\xi_3 = \frac{\text{volume of } P124}{\Omega^e}$$

$$\xi_4 = \frac{\text{volume of } P123}{\Omega^e}$$

Shape function: 4-node tetrahedral element



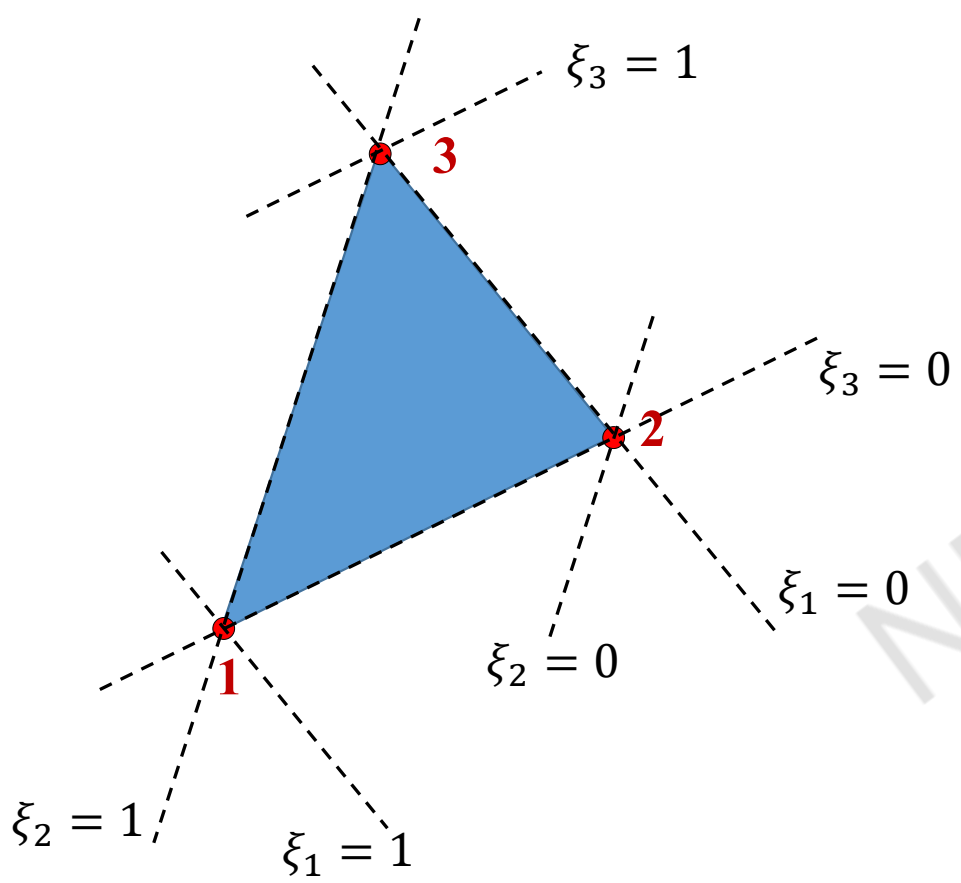
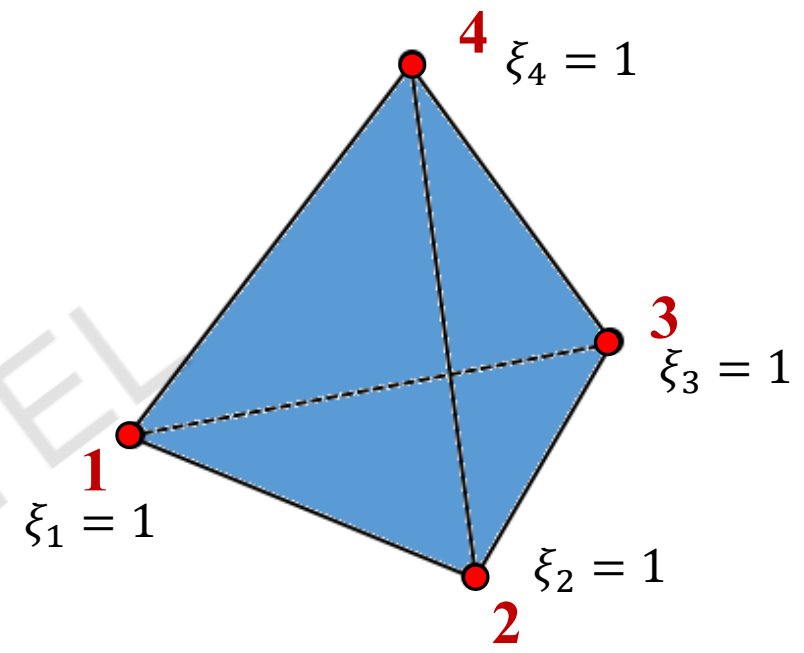
$$\phi_1^{4T} = \xi_1$$

$$\phi_2^{4T} = \xi_2$$

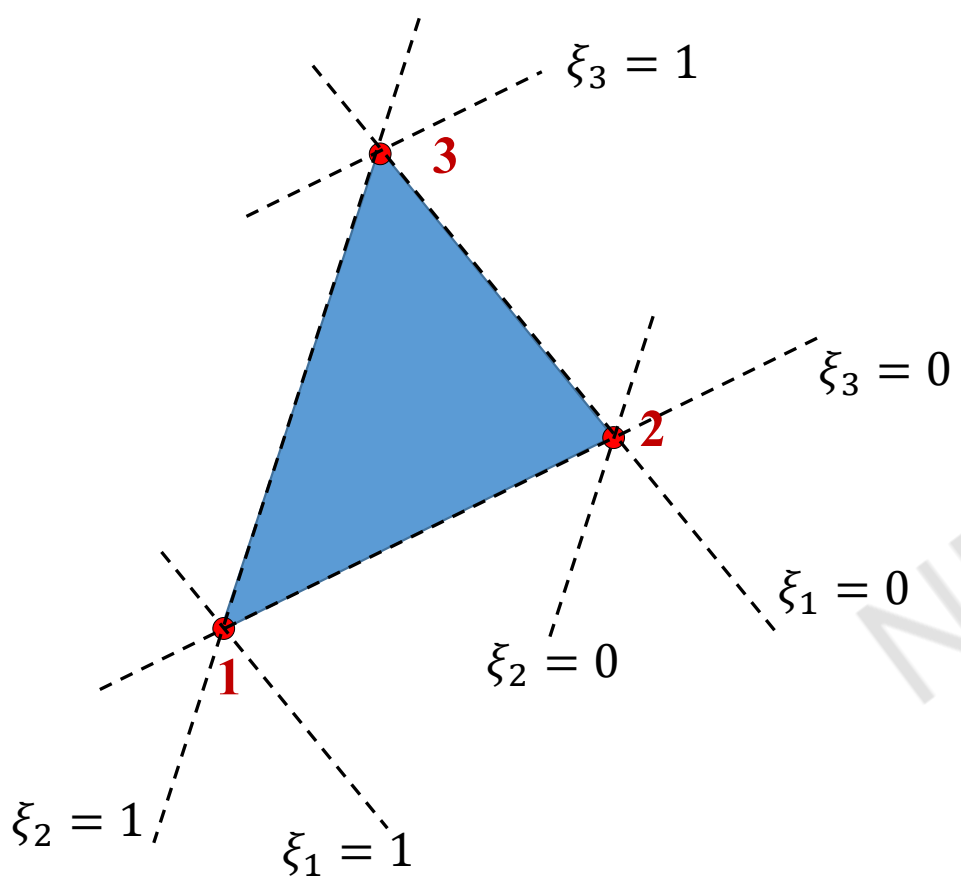
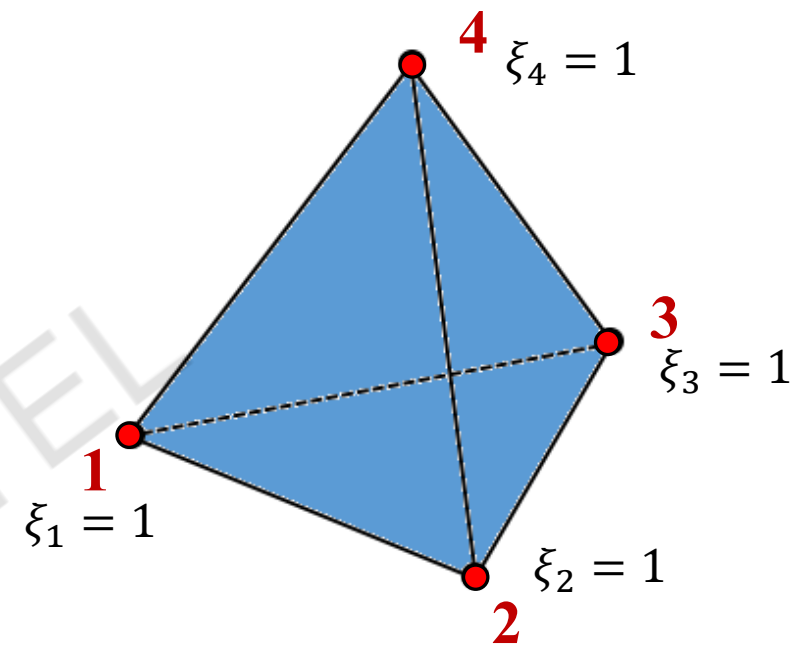
$$\phi_3^{4T} = \xi_3$$

$$\phi_4^{4T} = 1 - \xi_1 - \xi_2 - \xi_3$$

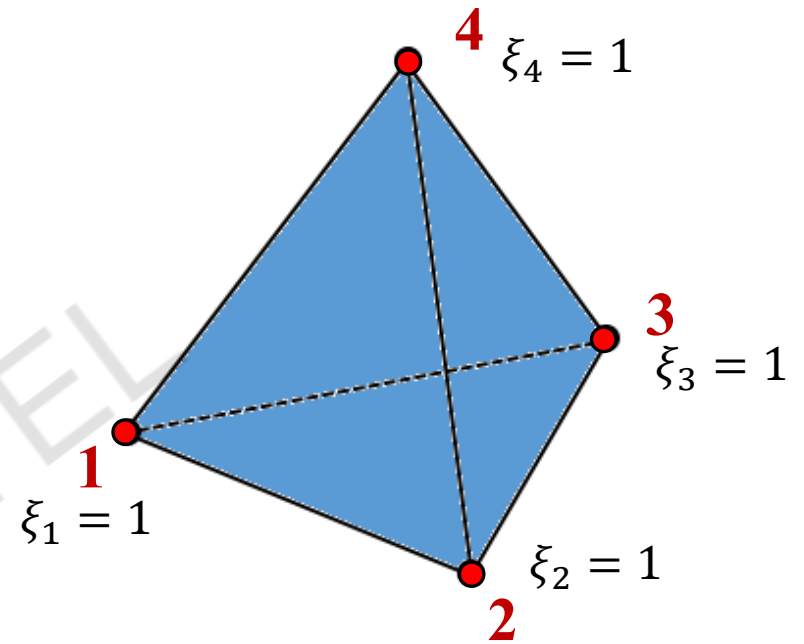
Recall: Area coordinate



Recall: Area coordinate

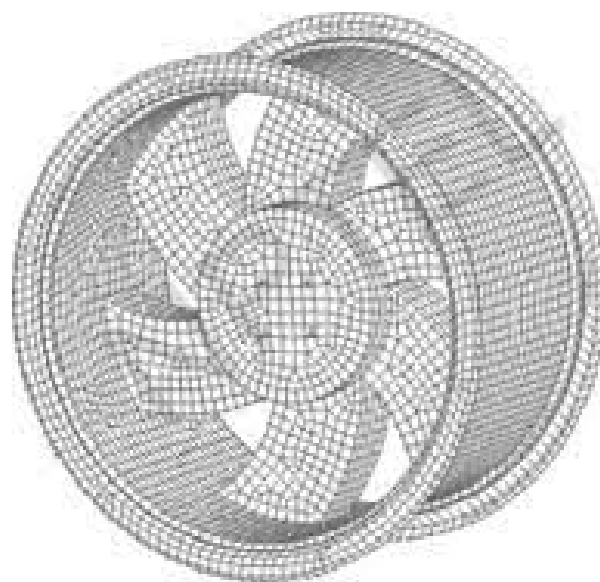
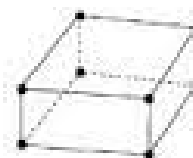


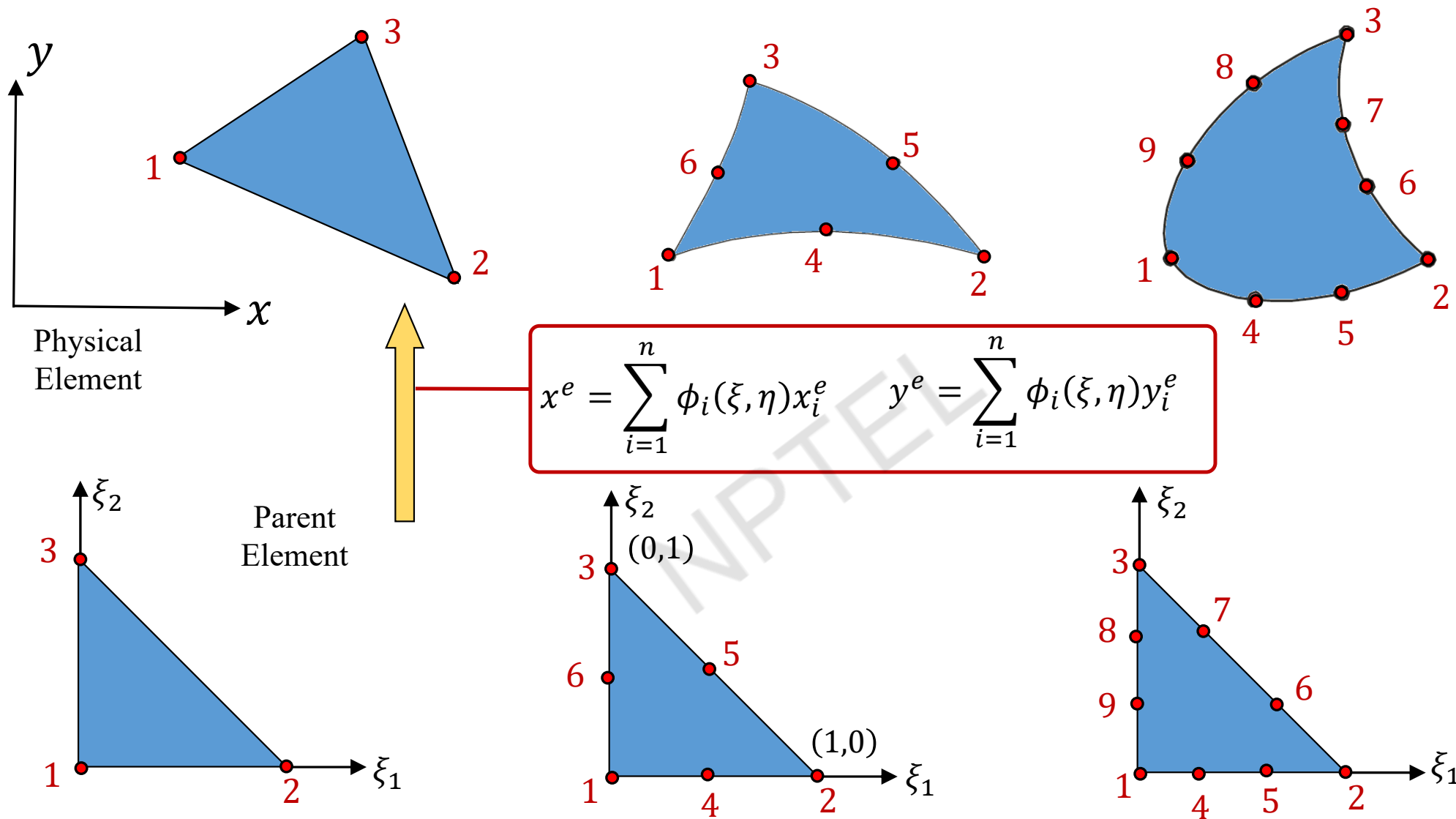
Recall: Area coordinate

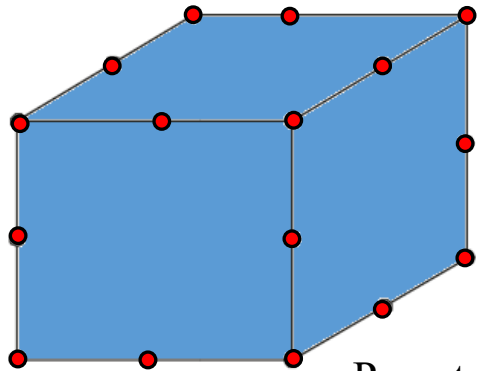


Shape function: 4-node tetrahedral element

NPTEL





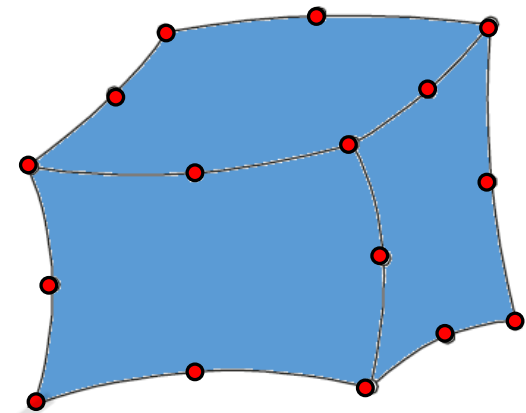


Parent
Element

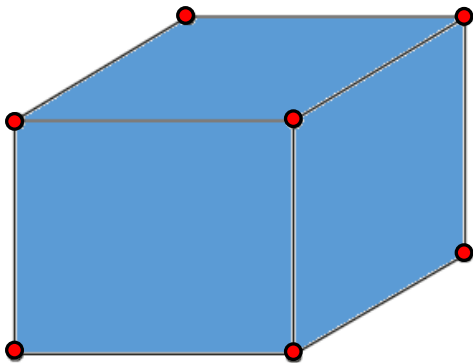
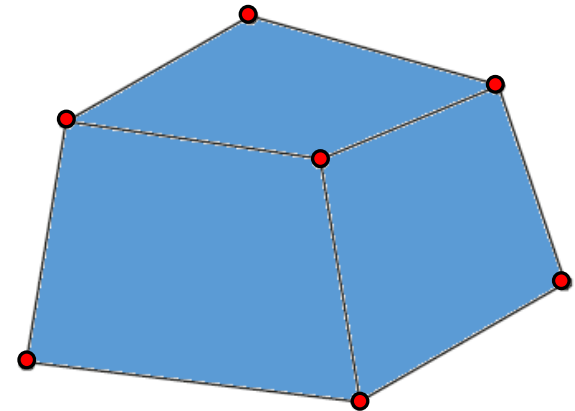
$$x^e = \sum_{i=1}^n \phi_i(\xi, \eta, \zeta) x_i^e$$

$$y^e = \sum_{i=1}^n \phi_i(\xi, \eta, \zeta) y_i^e$$

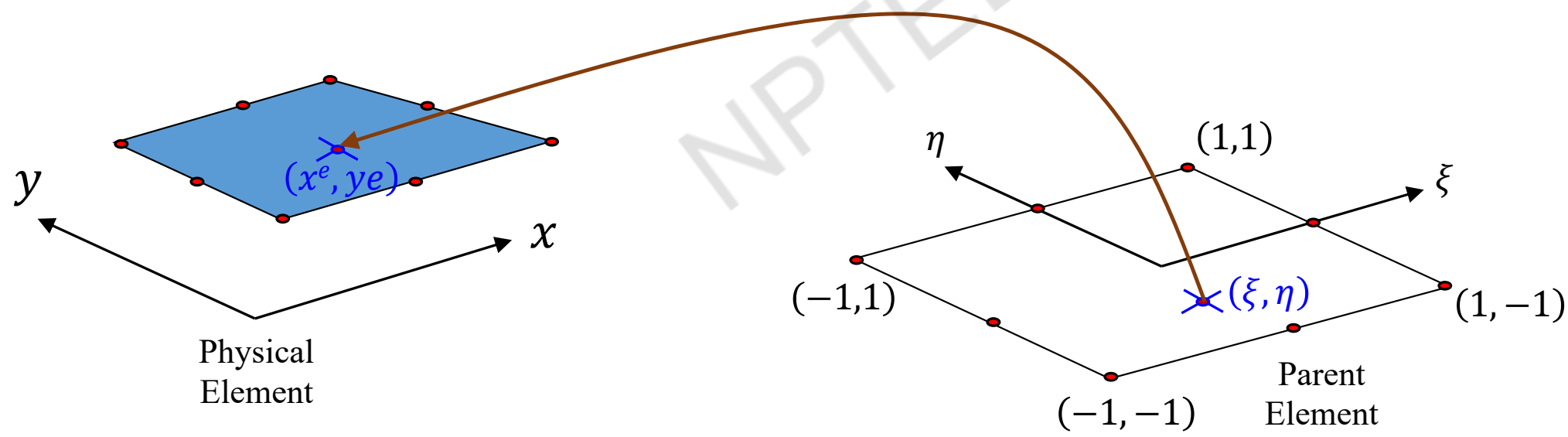
$$z^e = \sum_{i=1}^n \phi_i(\xi, \eta, \zeta) z_i^e$$



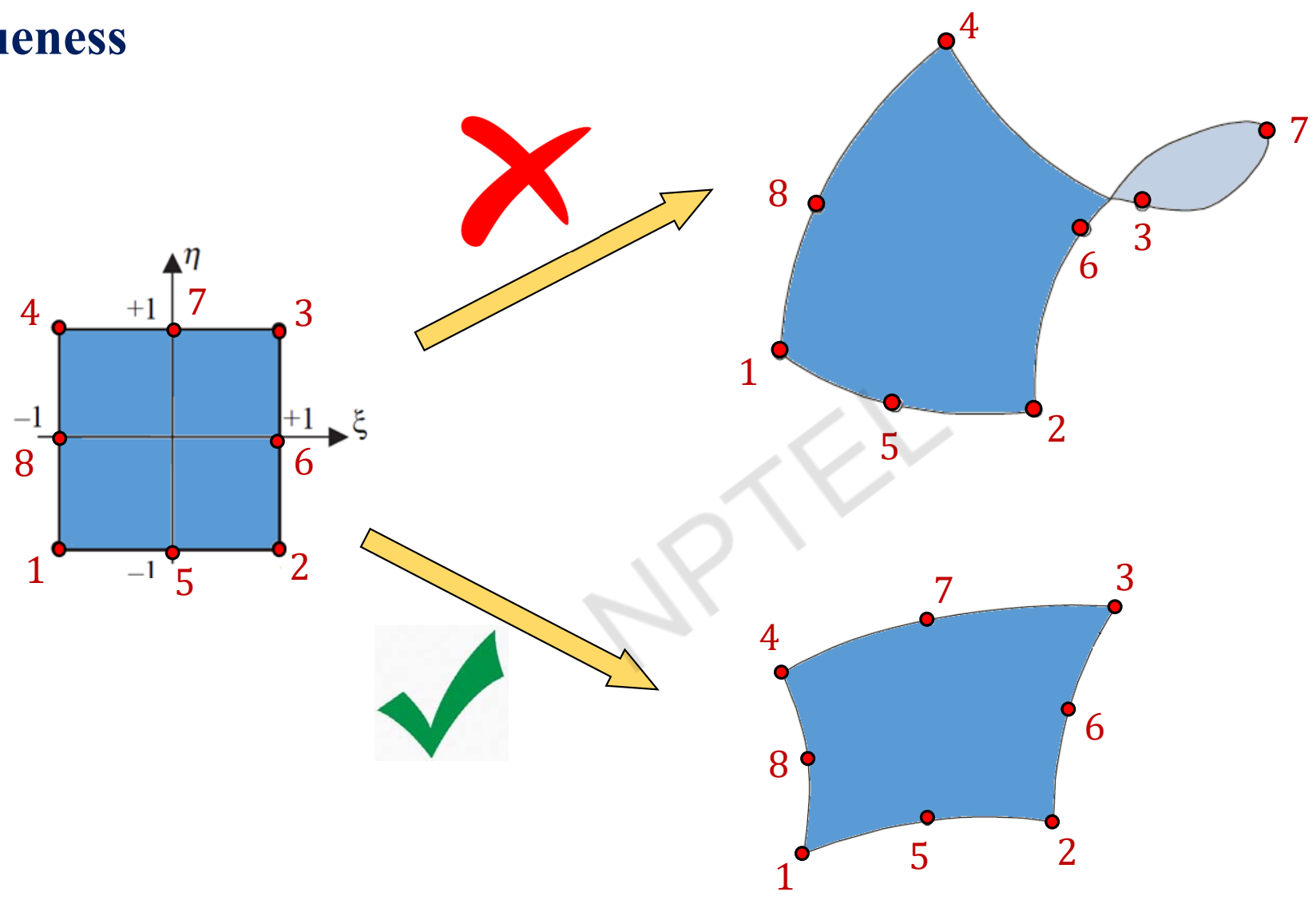
Physical
Element



Uniqueness



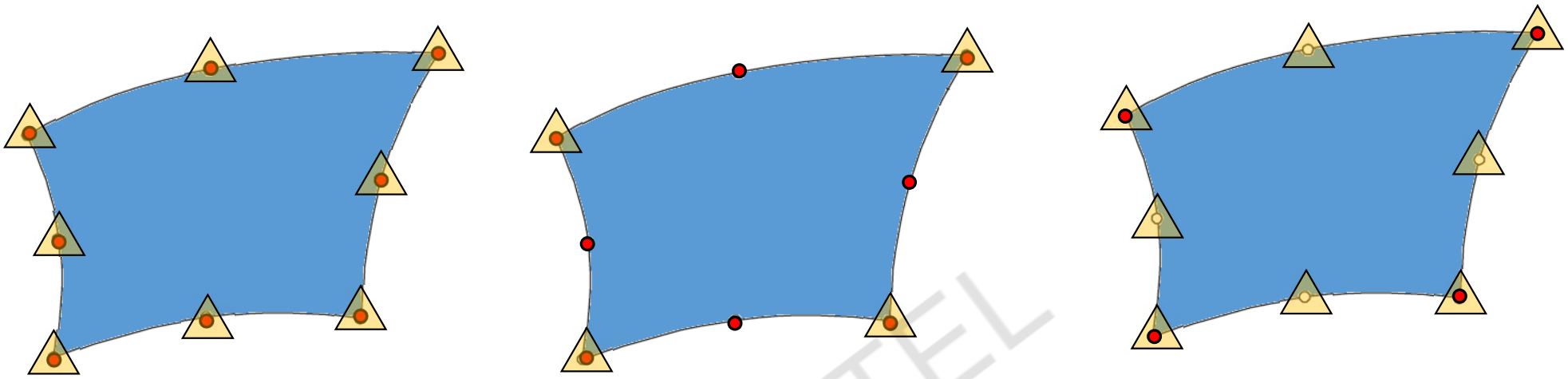
Uniqueness



Iso-parametric, Super-parametric and Sub-parametric

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Iso-parametric, Super-parametric and Sub-parametric

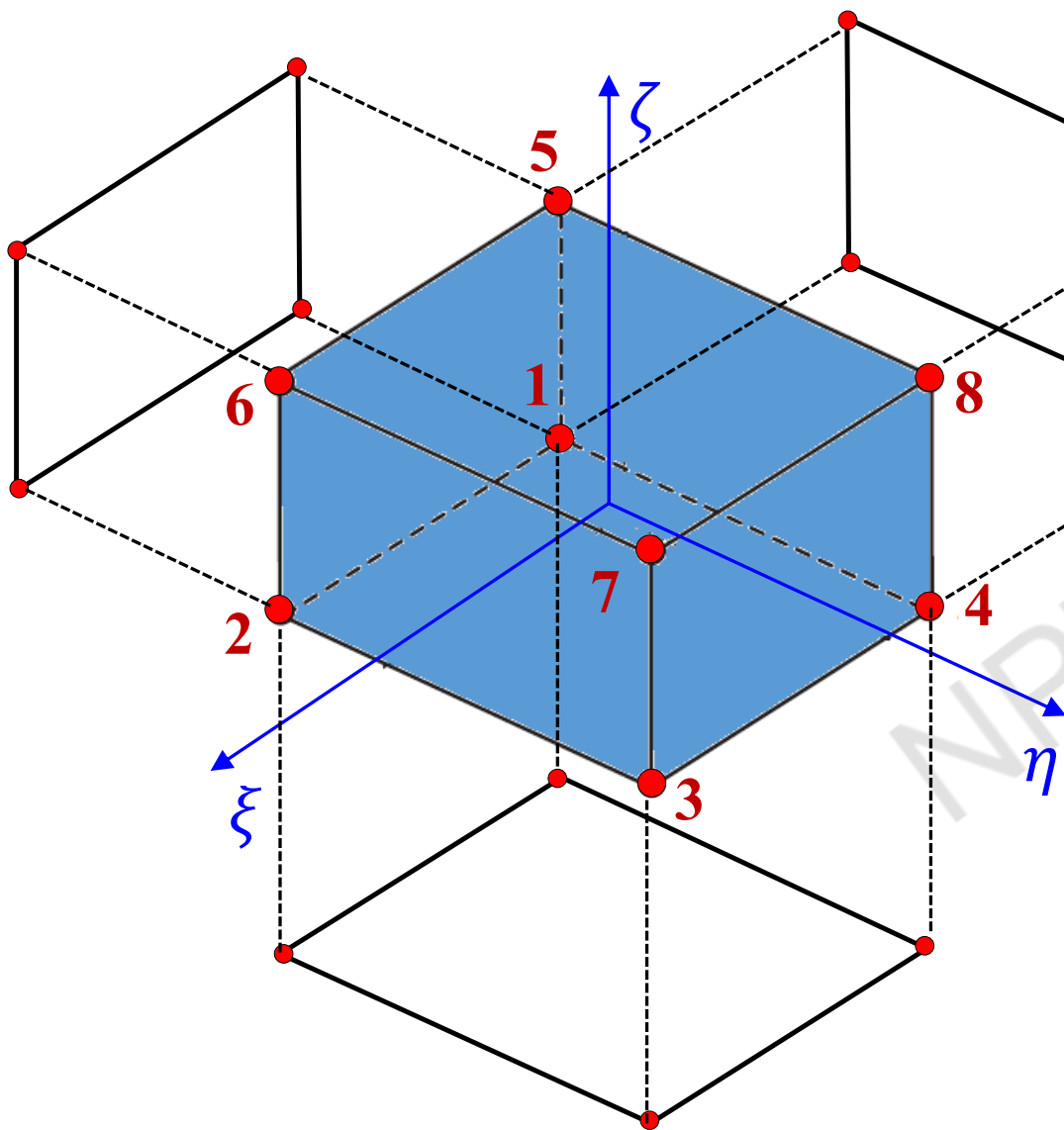


Iso-parametric

Super-parametric

Sub-parametric

- Nodes for approximating geometry
- △ Nodes for approximating field variables



$$\phi_L^{8H}(\xi, \eta, \zeta) = \phi_{[I,J,K]}^{8H}(\xi, \eta, \zeta)$$

$$= \phi_I^{2L}(\xi) \phi_J^{2L}(\eta) \phi_J^{2L}(\zeta)$$

L	I	J	K
1	1	1	1
2	2	1	1
3	2	2	1
4	1	2	1
5	1	1	2
6	2	1	2
7	2	2	2
8	1	2	2

Recall: Poisson equation

$$\sum_{j=1}^n \left\{ \int_{\Omega^e} \left[\frac{\partial \phi_i^e}{\partial x} \left(k \frac{\partial \phi_j^e}{\partial x} \right) + \frac{\partial \phi_i^e}{\partial y} \left(k \frac{\partial \phi_j^e}{\partial y} \right) \right] dx dy \right\} u_j^e = \int_{\Omega^e} f \phi_i^e dx dy + \oint_{\Gamma^e} \phi_i^e q_n ds$$

$$[K^e] \{u^e\} = \{f^e\} + \{Q^e\}$$

$$[K^e] = \int_{\Omega^e} \mathbf{B}^{eT} \mathbf{C} \mathbf{B}^e dx dy \quad \mathbf{C} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$\mathbf{B}^e = \begin{bmatrix} \frac{\partial \phi_1^e}{\partial x} & \frac{\partial \phi_2^e}{\partial x} & \cdots & \frac{\partial \phi_n^e}{\partial x} \\ \frac{\partial \phi_1^e}{\partial y} & \frac{\partial \phi_2^e}{\partial y} & \cdots & \frac{\partial \phi_n^e}{\partial y} \end{bmatrix}$$

n = number of nodes



**THANK
YOU !**



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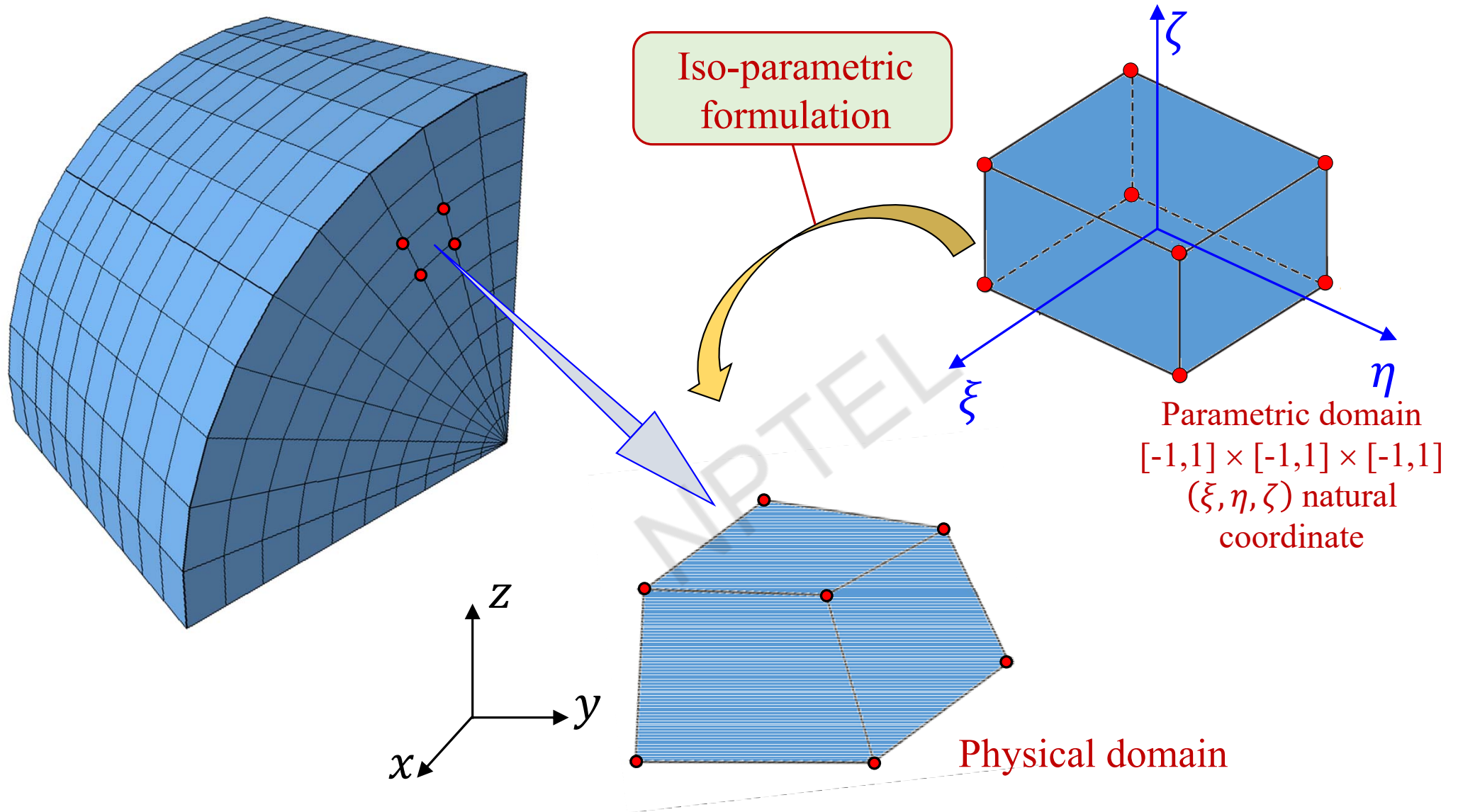
Finite Element Method

Week 12: Finite element in 3D

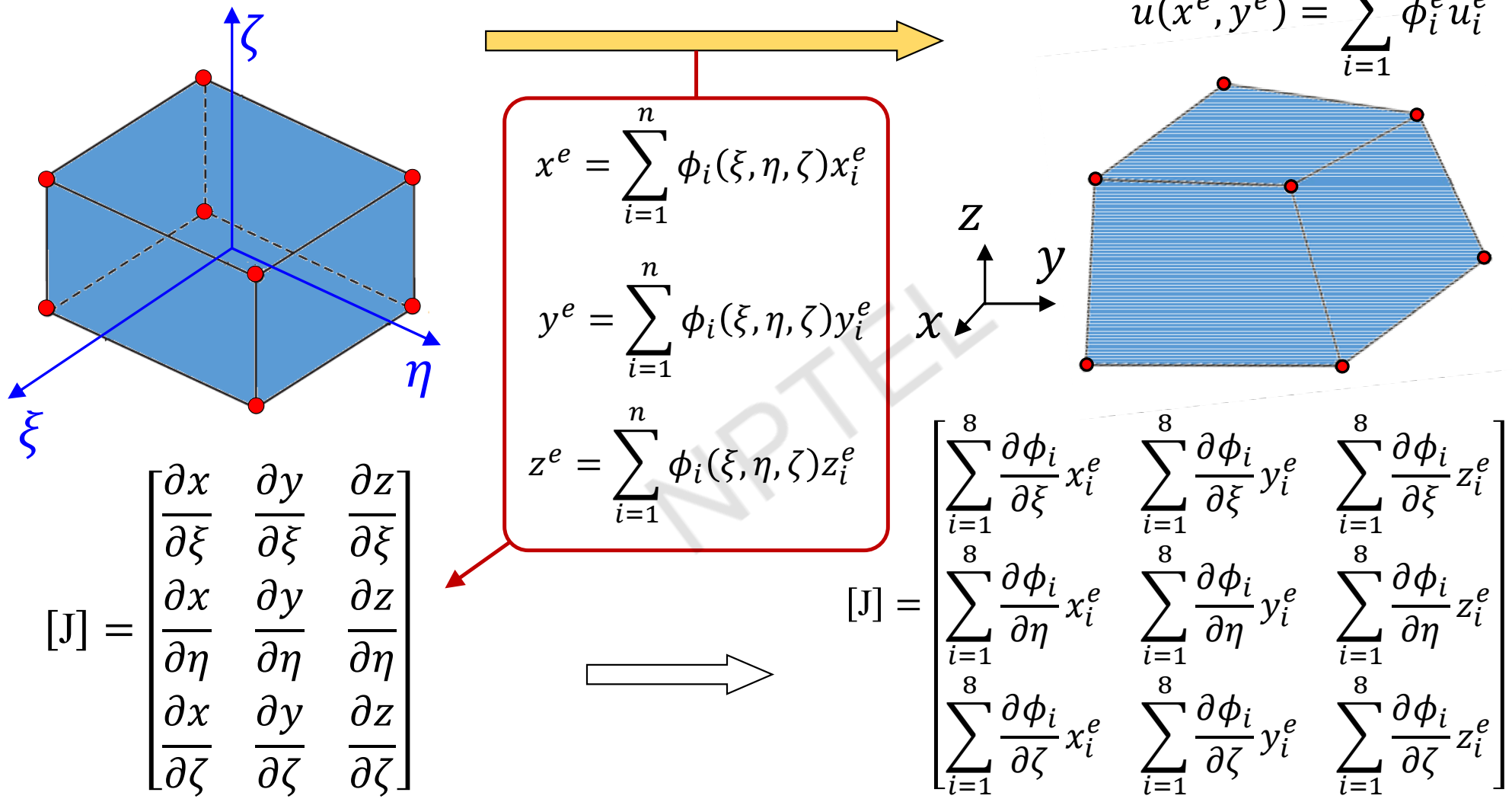
Lecture 59: Iso-parametric Formulation and Guass Quadrature

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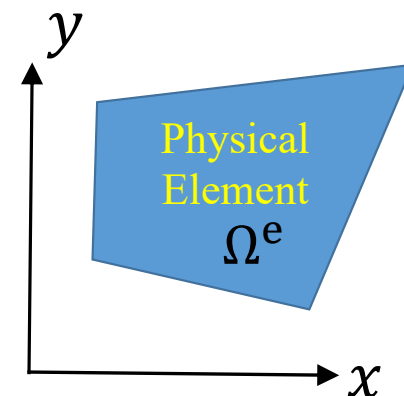
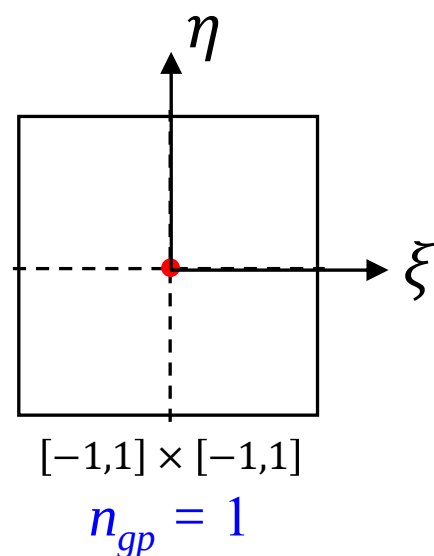
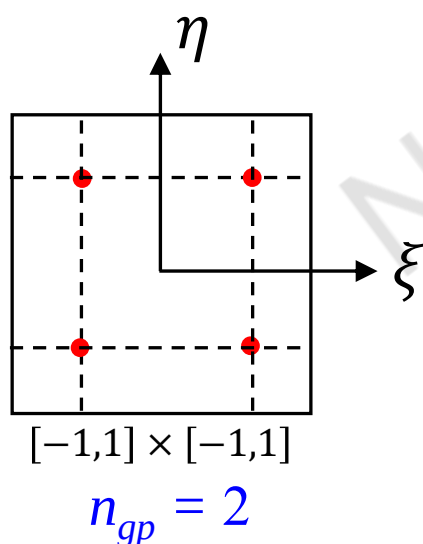
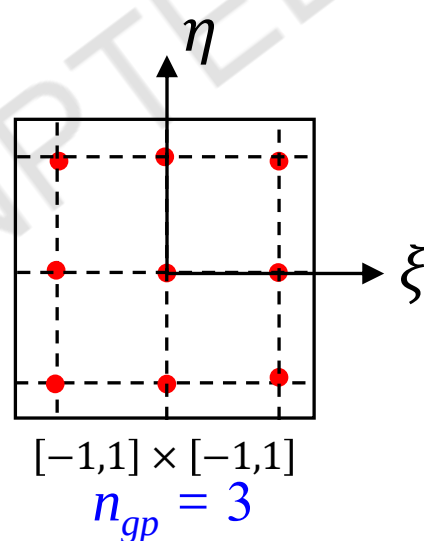
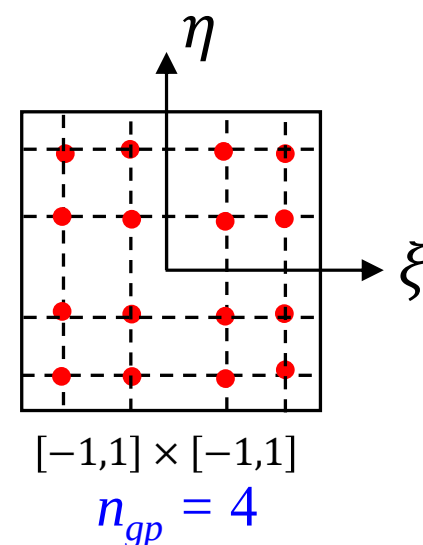
Iso-parametric Formulation



Recall: Integration (Gaussian Quadrature) Over Quadrilateral Domain

$$I = \int_{\Omega^e} f(x, y) d\Omega \Rightarrow I = \int_{\eta=-1}^1 \int_{\xi=-1}^1 |\mathbf{J}(\xi, \eta)| f(\xi, \eta) d\xi d\eta$$

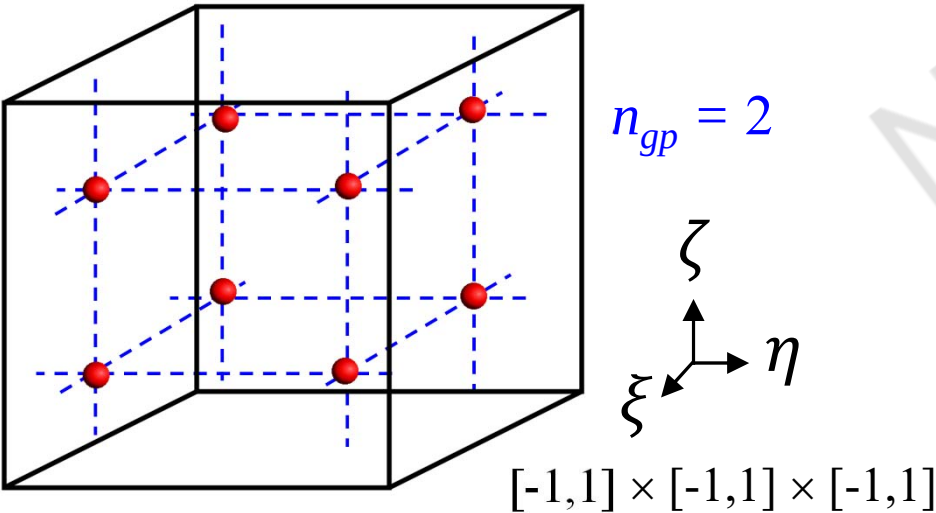
$$\Rightarrow I = \sum_{i=1}^{n_{gp}} \sum_{j=1}^{n_{gp}} W_i W_j |\mathbf{J}(\xi_i, \eta_j)| f(\xi_i, \eta_j)$$



Gaussian Quadrature Over Hexahedral (Brick) Domain

$$I = \int_{\Omega^e} f(x,y,z)d\Omega \Rightarrow I = \int_{\xi=-1}^1 \int_{\eta=-1}^1 \int_{\zeta=-1}^1 |\mathbf{J}(\xi,\eta,\zeta)| f(\xi,\eta,\zeta) d\xi d\eta d\zeta$$

$$\Rightarrow I = \sum_{i=1}^{n_{gp}} \sum_{j=1}^{n_{gp}} \sum_{k=1}^{n_{gp}} W_i W_j W_k |\mathbf{J}(\xi_i,\eta_j,\zeta_k)| f(\xi_i,\eta_j,\zeta_k)$$

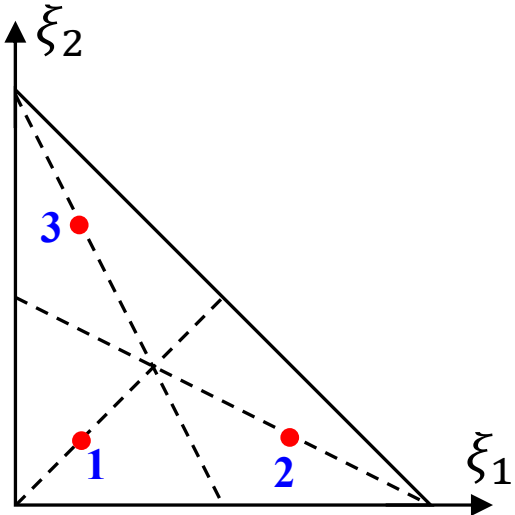
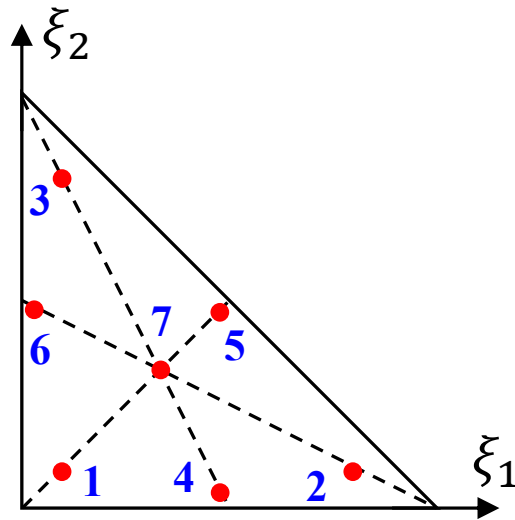


n	ξ_i, η_j, ζ_k	W_i, W_j, W_k
1	0.0	2.0
2	$\pm 1/\sqrt{3} = \pm 0.5773502692$	1.0
3	± 0.7745966692 0.0	0.555 555 5556 0.888 888 8889
4	± 0.8611363116 ± 0.3399810436	0.347 854 8451 0.652 145 1549

Recall: Integration Over Triangular Domain

$$I = \sum_{i=1}^n W_i |\mathbf{J}(\xi_{1i}, \xi_{2i})| f(\xi_{1i}, \xi_{2i})$$

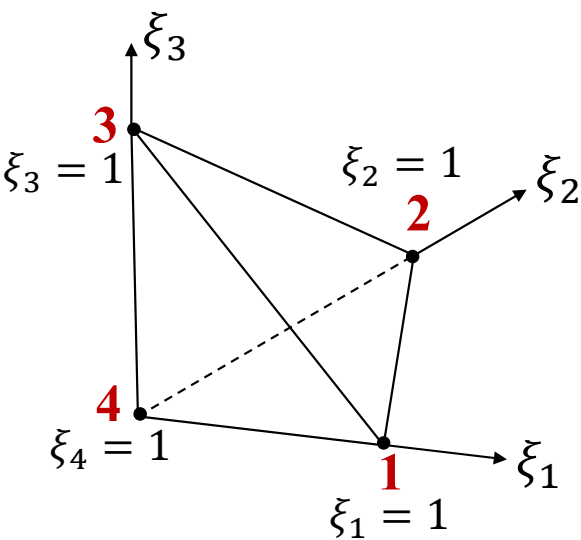
Integration order	ξ_{1i}	ξ_{2i}	W_i
Three-point	0.1 666 666 666	0.1 666 666 666	0.1 666 666 666
	0.6 666 666 666	0.1 666 666 666	0.1 666 666 666
	0.1 666 666 666	0.6 666 666 666	0.1 666 666 666
Seven-point	0.1 012 865 073	0.1 012 865 073	0.0 629 695 903
	0.7 974 269 853	0.1 012 865 073	0.0 629 695 903
	0.1 012 865 073	0.7 974 269 853	0.0 629 695 903
	0.4 701 420 641	0.0 597 158 717	0.0 661 970 764
	0.4 701 420 641	0.4 701 420 641	0.0 661 970 764
	0.0 597 158 717	0.4 701 420 641	0.0 661 970 764
	0.3 333 333 333	0.3 333 333 333	0.1125



Integration Tetrahedral Element

$$I = \sum_{i=1}^n W_i |\mathbf{J}(\xi_{1i}, \xi_{2i}, \xi_{3i})| f(\xi_{1i}, \xi_{2i}, \xi_{3i})$$

Integration order	ξ_{1i}	ξ_{2i}	ξ_{3i}	W_i
One-point	0.25	0.25	0.25	1
	0.58 541 020	0.13 819 660	0.13 819 660	0.25
Four-point	0.13 819 660	0.58 541 020	0.13 819 660	0.25
	0.13 819 660	0.13 819 660	0.58 541 020	0.25
	0.13 819 660	0.13 819 660	0.13 819 660	0.25
	0.25	0.25	0.25	-0.8
Five-point	1/3	1/6	1/6	0.45
	1/6	1/3	1/6	0.45
	1/6	1/6	1/3	0.45
	1/6	1/6	1/6	0.45
	1/6	1/6	1/6	0.45



Recall: Poisson's Equation in 2D $\nabla \cdot (k \nabla u) = -f$

Specific case ($a_{11} = a_{22} = k$, $a_{00} = a_{12} = a_{21} = 0$): Poisson equation

$$\sum_{j=1}^n \left\{ \int_{\Omega^e} \left[\frac{\partial \phi_i^e}{\partial x} \left(k \frac{\partial \phi_j^e}{\partial x} \right) + \frac{\partial \phi_i^e}{\partial y} \left(k \frac{\partial \phi_j^e}{\partial y} \right) \right] dx dy \right\} u_j^e = \int_{\Omega^e} f \phi_i^e dx dy + \oint_{\Gamma^e} \phi_i^e q_n ds$$

$$[K^e] \{u^e\} = \{f^e\} + \{Q^e\}$$

$$\{f^e\} = \int_{\Omega^e} f^e \phi_i^e dx dy \quad \{Q^e\} = \oint_{\Gamma^e} \phi_i^e q_n^e ds$$

$$[K^e] = \int_{\Omega^e} \mathbf{B}^{eT} \mathbf{C} \mathbf{B}^e dx dy \quad \mathbf{C} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$\mathbf{B}^e = \begin{bmatrix} \frac{\partial \phi_1^e}{\partial x} & \frac{\partial \phi_2^e}{\partial x} & \dots & \frac{\partial \phi_n^e}{\partial x} \\ \frac{\partial \phi_1^e}{\partial y} & \frac{\partial \phi_2^e}{\partial y} & \dots & \frac{\partial \phi_n^e}{\partial y} \end{bmatrix}$$

n = number of nodes

Element Formulation: Poisson's Equation

$$\nabla \cdot (k \nabla u) = -f$$

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\}$$

$$[K^e] = \int_{\Omega^e} \mathbf{B}^{eT} \mathbf{C} \mathbf{B}^e dx dy dz$$

$$\{f^e\} = \int_{\Omega^e} f^e \phi_i^e dx dy dz$$

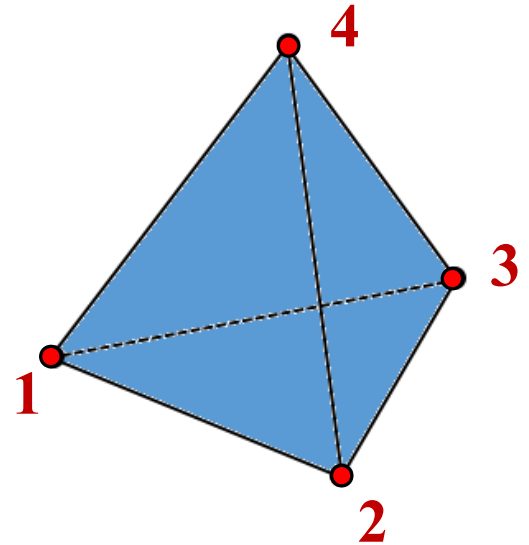
$$\{Q^e\} = \oint_{\Gamma^e} \phi_i^e q_n^e ds$$

$$\mathbf{C} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

$$\mathbf{B}^e = \begin{bmatrix} \frac{\partial \phi_1^e}{\partial x} & \frac{\partial \phi_2^e}{\partial x} & \frac{\partial \phi_3^e}{\partial x} & \frac{\partial \phi_4^e}{\partial x} \\ \frac{\partial \phi_1^e}{\partial y} & \frac{\partial \phi_2^e}{\partial y} & \frac{\partial \phi_3^e}{\partial y} & \frac{\partial \phi_4^e}{\partial y} \\ \frac{\partial \phi_1^e}{\partial z} & \frac{\partial \phi_2^e}{\partial z} & \frac{\partial \phi_3^e}{\partial z} & \frac{\partial \phi_4^e}{\partial z} \end{bmatrix}$$

$$\{f^e\} = \begin{Bmatrix} f_1^e \\ f_2^e \\ f_3^e \\ f_4^e \end{Bmatrix}$$

$$\{Q^e\} = \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$



$$\{u^e\} = \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \end{Bmatrix}$$

Element Formulation: Poisson's Equation

$$\nabla \cdot (k \nabla u) = -f$$

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\}$$

$$[K^e] = \int_{\Omega^e} \mathbf{B}^{eT} \mathbf{C} \mathbf{B}^e dx dy dz$$

$$\{f^e\} = \int_{\Omega^e} f^e \phi_i^e dx dy dz$$

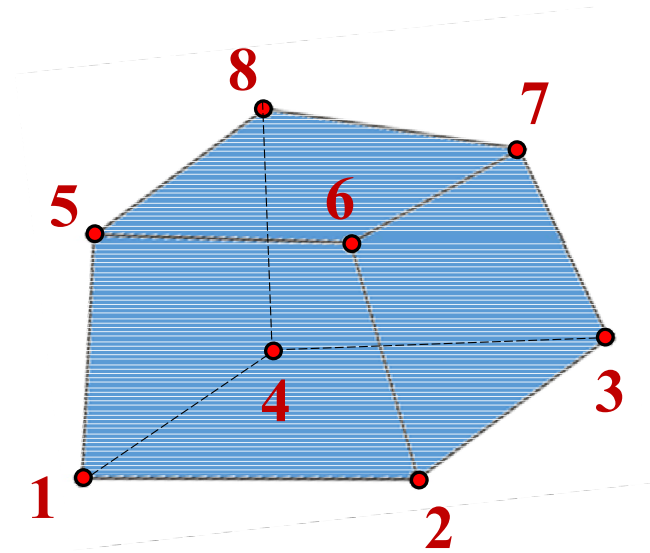
$$\{Q^e\} = \oint_{\Gamma^e} \phi_i^e q_n^e ds$$

$$\mathbf{C} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

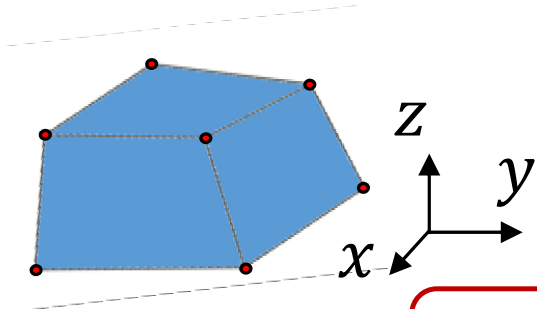
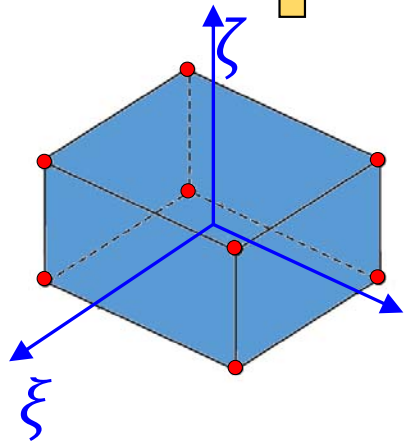
$$\mathbf{B}^e = \begin{bmatrix} \frac{\partial \phi_1^e}{\partial x} & \frac{\partial \phi_2^e}{\partial x} & \dots & \frac{\partial \phi_8^e}{\partial x} \\ \frac{\partial \phi_1^e}{\partial y} & \frac{\partial \phi_2^e}{\partial y} & \dots & \frac{\partial \phi_8^e}{\partial y} \\ \frac{\partial \phi_1^e}{\partial z} & \frac{\partial \phi_2^e}{\partial z} & \dots & \frac{\partial \phi_8^e}{\partial z} \end{bmatrix}$$

$$\{f^e\} = \begin{Bmatrix} f_1^e \\ f_2^e \\ \vdots \\ f_8^e \end{Bmatrix} \quad \{Q^e\} = \begin{Bmatrix} Q_1^e \\ Q_2^e \\ \vdots \\ Q_8^e \end{Bmatrix}$$

$$\{u^e\} = \begin{Bmatrix} u_1^e \\ u_2^e \\ \vdots \\ u_8^e \end{Bmatrix}$$



Element Formulation: Poisson's Equation



$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{Bmatrix}$$

$$\mathbf{B}^e(x, y, z) = \begin{bmatrix} \frac{\partial \phi_1^e}{\partial x} & \frac{\partial \phi_2^e}{\partial x} & \dots & \frac{\partial \phi_8^e}{\partial x} \\ \frac{\partial \phi_1^e}{\partial y} & \frac{\partial \phi_2^e}{\partial y} & \dots & \frac{\partial \phi_8^e}{\partial y} \\ \frac{\partial \phi_1^e}{\partial z} & \frac{\partial \phi_2^e}{\partial z} & \dots & \frac{\partial \phi_8^e}{\partial z} \end{bmatrix}$$

$$\mathbf{B}^e(\xi, \eta) = [J]^{-1} \begin{bmatrix} \frac{\partial \phi_1^e}{\partial x} & \frac{\partial \phi_2^e}{\partial x} & \dots & \frac{\partial \phi_8^e}{\partial x} \\ \frac{\partial \phi_1^e}{\partial y} & \frac{\partial \phi_2^e}{\partial y} & \dots & \frac{\partial \phi_8^e}{\partial y} \\ \frac{\partial \phi_1^e}{\partial z} & \frac{\partial \phi_2^e}{\partial z} & \dots & \frac{\partial \phi_8^e}{\partial z} \end{bmatrix}$$

$$[K^e] = \int_{\Omega^e} \mathbf{B}^{eT} \mathbf{C} \mathbf{B}^e dx dy dz$$

$$[K^e] = k \sum_{i=1}^{n_{gp}} \sum_{j=1}^{n_{gp}} \sum_{k=1}^{n_{gp}} W_i W_j W_k |\mathbf{J}(\xi_i, \eta_j, \zeta_j)| \mathbf{B}^{eT}(\xi_i, \eta_j, \zeta_j) \mathbf{B}^e(\xi_i, \eta_j, \zeta_j)$$



**THANK
YOU !**



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Finite Element Method

Week 12: Finite element in 3D

Lecture 60: Closure

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Recall: Plane Elasticity and Element Equation

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\}$$

$$[K^e] = \int_{\Omega^e} t_e \mathbf{B}^{eT} \mathbf{C} \mathbf{B}^e dx dy$$

$$\{f^e\} = \int_{\Omega^e} t_e \begin{bmatrix} \phi_1^e & 0 \\ 0 & \phi_1^e \\ \phi_2^e & 0 \\ 0 & \phi_2^e \\ \vdots & \vdots \\ \phi_n^e & 0 \\ 0 & \phi_n^e \end{bmatrix} \begin{bmatrix} f_x^e \\ f_y^e \end{bmatrix} dx dy$$

$$\mathbf{B}^e(x, y) = \begin{bmatrix} \frac{\partial \phi_1^e}{\partial x} & 0 & \frac{\partial \phi_2^e}{\partial x} & 0 & \dots & \frac{\partial \phi_n^e}{\partial x} & 0 \\ 0 & \frac{\partial \phi_1^e}{\partial y} & 0 & \frac{\partial \phi_2^e}{\partial y} & \dots & 0 & \frac{\partial \phi_n^e}{\partial y} \\ \frac{\partial \phi_1^e}{\partial y} & \frac{\partial \phi_1^e}{\partial x} & \frac{\partial \phi_2^e}{\partial y} & \frac{\partial \phi_2^e}{\partial x} & \dots & \frac{\partial \phi_n^e}{\partial y} & \frac{\partial \phi_n^e}{\partial x} \end{bmatrix}$$

$$\{Q^e\} = \int_{\Gamma^e} t_e \begin{bmatrix} \phi_1^e & 0 \\ 0 & \phi_1^e \\ \phi_2^e & 0 \\ 0 & \phi_2^e \\ \vdots & \vdots \\ \phi_n^e & 0 \\ 0 & \phi_n^e \end{bmatrix} \begin{bmatrix} T_x^e \\ T_y^e \end{bmatrix} ds$$

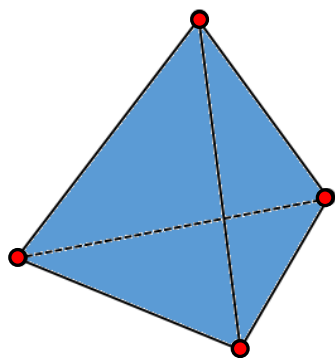
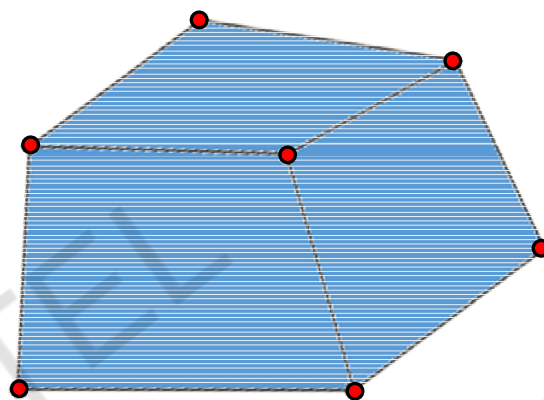
Plane Stress

$$\mathbf{C} = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1 - \nu)/2 \end{bmatrix}$$

Plane Strain

$$\mathbf{C} = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu & 0 \\ \nu & 1 - \nu & 0 \\ 0 & 0 & (1 - 2\nu)/2 \end{bmatrix}$$

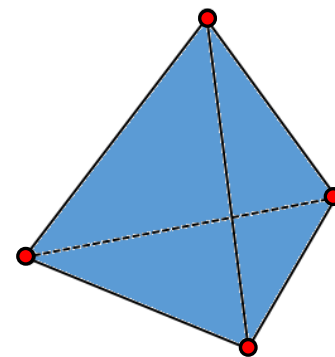
$$[\mathbf{C}] = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1 - \nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1 - \nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1 - 2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1 - 2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1 - 2\nu}{2} \end{bmatrix}$$



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$$[K^e] = \int_{\Omega^e} t_e \mathbf{B}^{eT} \mathbf{C} \mathbf{B}^e dx dy$$

$$\mathbf{B}^e(x, y) = \begin{bmatrix} \frac{\partial \phi_1^e}{\partial x} & 0 & 0 \\ 0 & \frac{\partial \phi_1^e}{\partial y} & 0 \\ 0 & 0 & \frac{\partial \phi_1^e}{\partial z} \\ \frac{\partial \phi_1^e}{\partial y} & \frac{\partial \phi_1^e}{\partial x} & 0 \\ 0 & \frac{\partial \phi_1^e}{\partial z} & \frac{\partial \phi_1^e}{\partial y} \\ \frac{\partial \phi_1^e}{\partial z} & 0 & \frac{\partial \phi_1^e}{\partial x} \end{bmatrix}$$





**THANK
YOU !**